RESEARCH STATEMENT

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1. Introduction

In my research, I study modern aspects of classical computability theory, with particular focus on its interactions with reverse mathematics and algorithmic randomness. My projects include defining almost-everywhere computability, studying the large-scale structure of the Zoo of reverse-mathematical principles, and investigating fine divisions in reverse mathematics.

As a brief reminder, we say that a set $A \subseteq \mathbb{N}$ is computable if there is an algorithm for deciding whether an arbitrary n belongs to A. In general, classical computability focuses on non-computable subsets of $\mathbb{N}$ and various relations between them, particularly Turing reducibility. Given two non-computable sets $A, B \subseteq \mathbb{N}$, we say A is Turing reducible to B if A is computable with reference to an *oracle* for B, which can provide answers to questions of the form "Is $x \in B$?"; if this is so, then B is "at least as hard to compute" as A. (For a more thorough summary of the background of classical computability, please see Appendix A.)

2. Almost-Everywhere Computability

Computability theorists typically consider a set A to be nearly computable if it is somehow close to the minimum Turing degree (that of the computable sets); that is, if there is an algorithm that correctly computes membership in A given access to weakly non-computable information. In this project, I focus on sets that are almost computable — but in an atypical sense. Rather than working with sets that are computable with a weak oracle, I consider sets A where we can *usually* compute whether $n \in A$, with no outside help; that is, sets that are almost-everywhere computable.

Since we are working with subsets of $\mathbb{N}$, we need to define what "usually" means in this context; we have no canonical definition for "with probability 1". The standard approach would be to adopt asymptotic density as a pseudomeasure, and consider sets to be large if they have density 1. This approach, originated by Jockusch and Schupp [14] and since investigated by many others (including myself), yields a weak notion of computability that trades off compatibility with classical tools for ease of analysis. As an alternative, I developed a stricter pseudomeasure on $\mathbb{N}$ (called *intrinsic density*) that is compatible with the standard tools of computability; we say that a set with intrinsic density 0 is *effectively negligible.* Using this pseudomeasure, we can then say that a set is *intrinsically* computable with density 1, though this admits four possibly-distinct definitions. Much remains to be done in analyzing these definitions; however, this approach has already shown stronger connections to classical results of computability, fitting naturally into the context of the field as a whole while suggesting still more connections between computability and algorithmic randomness.

I also plan to apply this analysis in complexity theory, with hope that it will help characterize the surprising practical efficiency of certain algorithms (e.g., SAT solvers) with known poor worst-case performance. It seems unlikely that such

Date: September 25, 2017.

algorithms have high performance on a set of intrinsic density 1, but if it were to be true, this might prove significant in the study of NP-complete problems.

In the following subsections, I discuss in greater detail each of the three subprojects within this heading: asymptotic computability (modulo sets of density 0), computability modulo effectively negligible error, and investigation of intrinsic density itself as a property of some computability-theoretic interest.

2.1. **Asymptotic computability.** As discussed above, this project is ultimately motivated by a simple question: what does it mean to say that a set is "computable almost everywhere"? Since there is no uniform probability measure on the natural numbers, there can be no canonical answer. Instead, the answer depends on how one chooses to define a negligible subset of $\mathbb{N}$.

Kapovich, Myasnikov, Schupp, and Shpilrain [15] and Jockusch and Schupp [14] began by borrowing a standard answer from number theory. They defined a set A to be negligible if $\lim_{n\to\infty} \frac{|A\restriction n|}{n} = 0$ — that is, if the set has *asymptotic density* 0. Applying this definition, Jockusch and Schupp defined coarse and generic computability as computability modulo a negligible set, varying by precisely how such a computation is allowed to fail in the neglected cases. They also obtained notions of relative co-negligible computability by extending in the traditional manner, though due to complications in the oracle, they were forced to make further changes to obtain a transitive reducibility. This led to immediate (and apparently hard) questions on the structure of the coarse and generic degrees. Even basic questions resisted initial analysis, such as whether there is a non-zero degree comparable to all other degrees.

Downey, Jockusch, and Schupp [7] then asked whether either structure contains minimal pairs, as in the Turing degrees, but obtained only partial results for relative generic computability. Despite significant work on the question by Igusa [12] (refuting minimal pairs for relative generic computability) and Hirschfeldt, Jockusch, Kuyper, and Schupp [9] (constructing minimal pairs for coarse reducibility), the question remains open for generic reducibility.

In forthcoming work, Hirschfeldt, Jockusch, and I [5] provide more insight into this problem. In contrast to Igusa's result, we show that it is *nearly* possible to build minimal pairs for generic reducibility, and only a near-miss for computability, by avoiding upper cones:

Theorem 2.1. *If A is not generically computable, the class of sets in $2^{\mathbb{N}}$ which compute a generic description of A has measure 0.*

That is, non-trivial upper cones for generic computation have measure 0 and so are easily avoided. For coarse reductions, as well as two new complementary notions of co-density-0 computability and reduction explored in this paper, analogous results hold — and for coarse reducibility (as well as one of the new reductions), the result transfers to give a construction of a minimal pair. Unfortunately, the transfer appears to fail for the other two cases (including generic reducibility), leaving the question open for future investigation.

Future work. Several precise relations between the notions of asymptotic computation remain to be determined; some suggested definitions may even be equivalent to combinations of already-studied properties, at least in certain contexts. I intend to map the Turing degrees exhibiting these combinations (such as sets that are generically, but not coarsely, computable), which would be useful in characterizing these relations.

2.2. **Computation modulo effectively negligible sets.** However, much of my work takes a different approach to negligibility than that of Jockusch and Schupp.

To preserve more of computability theory, I have adopted a stricter goal of *effective negligibility*, shifting to a variant pseudomeasure. For instance, though the powers of 2 have density 0, it seems unnatural to say that this set is *effectively* negligible; as one can trivially enumerate powers of 2, an algorithm with access to an oracle might end up asking exclusively about members of this set. Instead, I say that a set has *intrinsic density* 0 if its images under computable permutations of $\mathbb{N}$ all have density 0. Furthermore, generic computability is extremely sensitive to the coding of its inputs, to the point where some forms of the halting problem can be shown to be generically computable; intrinsic density is designed to prevent this sensitivity.

Treating sets with intrinsic density 0 as effectively negligible, I have proposed four possible definitions of intrinsic generic computability, differing by the degree of uniformity each requires. In my first paper [2], I extended Rice's Theorem to intrinsic generic computability in the following form:

Theorem 2.2. *There is a computable permutation under which the image of no non-trivial index set is generically computable.*

In this context, a set A is said to be a *non-trivial index set* if $A \neq \emptyset$, $A \neq \mathbb{N}$, and whenever i and j are indices of equivalent Turing machines, $i \in A$ iff $j \in A$. This proves that, regardless of one's choice of definition,

Corollary 2.3. *No non-trivial index set, nor anything 1-equivalent to a non-trivial index set, is intrinsically generically computable.*

In particular, the halting problem is not intrinsically generically decidable.

Future work. In research still to come, I plan to explore the possible choices of uniformity in defining intrinsic generic computability; my current hypothesis is that the various feasible definitions will prove not to be equivalent, and that one will emerge as the most natural definition. In parallel, I will develop a theory of intrinsic generic *reduction* between sets, with intentions to apply the result to the complexity-theoretic project discussed in this section's introduction.

Further, there are three other notions of co-density-0 computability, two of which are compatible with the restrictions needed to develop a corresponding notion of intrinsic computation. Some preliminary results, not yet organized for publication, suggest that intrinsic *coarse* computability will be quite different from intrinsic generic computability, and interesting in its own right.

2.3. **Intrinsic density.** Intrinsic density itself is also a rich object of study, with surprising connections to both algorithmic randomness and classical computability.

Computability theorists have studied sparse sets since Post first began defining "thinness" properties in the course of his program to produce a non-computable incomplete set, building a complex hierarchy of forms of sparsity known as immunity properties. My notion of intrinsic density 0 (ID0) strengthens Post's definition of immunity (his weakest notion of thinness) and is in fact a new and natural immunity property in the classical sense; it appears to be the first that is completely incomparable to the "hyperimmunity" section of the standard hierarchy. In my first paper [2], I determined all implications and non-implications between ID0 and the classical immunity properties. The proofs involved some interesting constructions, particularly those demonstrating the independence of ID0 from the variants of hyperimmunity; one in particular required an *a priori* Δ^0_3-construction to be carried out below $\emptyset'$.

Analogously, intrinsic density 1/2 provides a new notion of stochasticity, connected to two previously-defined notions of randomness (permutation and injection randomness). This link intrinsic density provides between immunity and stochasticity highlights a philosophical feature of immunity that is often neglected: a set is said to be immune precisely if it is sufficiently difficult to predict which elements belong to it.

This interpretation of immunity, at least in the case of intrinsic density 0, is quite exact. In a forthcoming paper [1], I show that in the sense of Turing degrees, it is precisely as difficult to compute a set with defined intrinsic density as it is to compute a function that agrees at most finitely often with any computable function — that is, a function whose values cannot be infinitely often predicted. Using a theorem of Kjos-Hanssen, Merkle, and Stephan [16] to put this result into classical (degree-theoretic) terms, we can say:

Theorem 2.4. *A Turing degree a contains a set with defined intrinsic density iff a is high (computes a dominating function) or DNC (computes a function f such that $f(e)$ is never equal to $\varphi_e(e)$).*

In fact, this equivalence is effective enough to hold in reverse mathematics:

Theorem 2.5. *The existence of sets with intrinsic density 0 is, over* RCA_0*, equivalent to the disjunction of the principles* DNR *and* DOM*; that is, the existence of DNC or dominating functions.*

Future work. Few immediately-obvious questions remain regarding sets with intrinsic density 0; therefore, I will next focus on sets with intermediate intrinsic density (e.g., intrinsic density $\frac{1}{2}$), and resolving some possibilities regarding details of sets with intrinsic density 1.

Sets with intrinsic density $\frac{1}{2}$, as mentioned above, are those exhibiting a certain amount of stochasticity. As such, their Turing degrees are closely related to those exhibiting weak forms of randomness; for instance, every Schnorr random has intrinsic density $\frac{1}{2}$, so such sets exist in every high or 1-random degree. On the other hand, our only lower bound shows that sets with well-defined intrinsic density have either high or DNC degree; as not every DNC degree is 1-random, this leaves a gap that I will investigate in the future.

As for sets with intrinsic density 1, a few classical questions remain open, particularly regarding the properties of c.e. sets with intrinsic density 1. These are largely minor details in the characterization of such sets as those with a simplicity property (co-immunity restricted to c.e. sets), which do not quite follow from my current proofs characterizing intrinsic density 0 as an immunity property; additional work has already begun to close the gap, so I expect this to follow in a reasonable amount of time.

In addition, I have recently begun to generalize the results of Downey, Jockusch, and Schupp on sets with density 1 to the case of intrinsic density 1. For example, Jockusch and Schupp [14] demonstrated the existence of a c.e. set with density 1 with no density-1 computable subset; with Downey, they then proved that such sets exist in precisely the non-low c.e. Turing degrees (those with jump strictly above $\mathbf{0}'$) [7]. In joint work between Cholak, Igusa, and myself, we have tentatively constructed a c.e. set with *intrinsic* density 1 that has no density-1 computable subset, though this result is preliminary and has not yet been reviewed. In the future, I plan to characterize the Turing degrees of such sets.

3. Large-Scale Structure in the Reverse-Mathematical Zoo

Over the last few decades, a new program in mathematical logic has emerged; founded by Friedman [8] and advanced by Simpson, reverse mathematics determines precisely which axioms are necessary to prove theorems of ordinary mathematics, representing objects by sets of natural numbers and working over a weak subsystem of second-order arithmetic. Due to well-studied connections between computability and definability, this base system (RCA_0) corresponds to the "Recursive Comprehension Axiom": every computable set exists. Within this program, typical results have two forms: "Principle A is true in every model of RCA_0 in which Principle B holds," or its negation. If two principles imply each other in this way, we say they are equivalent over RCA_0. Early in the development of the field, Friedman, Simpson, and others noticed that the vast majority of theorems successfully analyzed could be proven equivalent to one of the "Big Five" subsystems, all linearly ordered by strength, each of which could be taken as corresponding to a permissible form of mathematical argument (constructive, finistic, etc.). (For additional background on reverse math and the Big Five systems, please see Appendix B.)

By the late 1990's, on the other hand, Seetapun and Slaman [20] had analyzed Ramsey's theorem for pairs and two colors:

Definition 3.1 (RT^2_2)**.** For every 2-coloring of unordered pairs in $\mathbb{N}$, there is an infinite subset within which all pairs have the same color.

In a celebrated theorem, they showed that RT^2_2 is strictly weaker than ACA_0, and so (combining this with a computability-theoretic result of Jockusch [13]) could not be equivalent to any of the Big Five systems. In fact, as we now know [17], RT^2_2 falls outside the linear order of the Big Five, showing that the principles studied in reverse mathematics exhibit a non-trivial large-scale structure. (This explanation is ahistorical, as RT^2_2 was not the first principle proven to lie outside the linear Big Five; after Seetapun and Slaman, several strictly-weaker principles were found that were more easily proven incomparable to one of the "Big Five" principles. However, these earlier exceptions have sometimes been criticized for being somewhat contrived, rather than theorems of "ordinary mathematics".)

These results have been followed by an explosion of discoveries, revealing a complex taxonomy of principles sometimes referred to as the Reverse-Mathematical Zoo. (For perspective, Figure 1 shows a simplified diagram of a subset of this Zoo, chosen for its relative comprehensibility.) Several mathematicians studying this have observed that the section between RCA_0 and ACA_0 can be seen as dividing into three parts, or branches, along with one exceptional principle: a randomness branch (containing principles that state that sufficiently [Martin-Löf–]random objects exist), a genericity branch (stating that sufficiently [Cohen–]generic objects exist), a Ramsey-theoretic branch (stating that sufficiently large objects must contain some form of order), and the exceptional subsystem WKL_0 (RCA_0, augmented by Weak König's Lemma[1]). These correspond roughly to three forms of proof: appeal to typicality (if an object is common, it must exist; this is similar to the probablistic method of Erdős), existence of unavoidable structure (as in Ramsey theory), and arguments by compactness.

These subjective branches continue to elude rigorous definition; however, recent results have begun to suggest that proper analysis may be possible. In a forthcoming paper [3], an international collaboration (Bienvenu, Dzhafarov, Patey, Shafer, Solomon, Westrick, and myself) will place upper bounds on the reverse-mathematical strength of randomness- and genericity-existence principles, and in fact of any principle describing a problem for which solutions are "not atypical" in some rigorous sense.[2] For principles where this approach is not applicable, we have developed subtle variants on this metatheorem based on more general properties.

[1](WKL) Every infinite binary-branching tree contains an infinite path.

[2]The bound applies for any notion of typicality that satisfies both a weak analogue of Fubini's theorem and closure under countable intersection.

Future work. A great deal remains to be explained in the large-scale structure of the Zoo. I will continue to seek rigorous definition of and bounds on the Ramsey-theoretic branch (though this branch is known to extend through ACA_0, the next-higher system of the Big Five). However, a larger question remains: are these the only significant natural branches of the Zoo, or might there be other forms of proof not easily categorized by the Big Five? I expect this will be a focus of my research for some time to come.

Also, obtaining a large-scale perspective on this subject requires one to keep track of an enormously active field of research, as the big picture depends on hundreds of individual results. As one might expect, computer assistance makes this far more feasible. I currently maintain the RM Zoo, a public computer-aided database for analysis of results in reverse mathematics; it provides an authoritative bibliography for the subject, an inference system capable of extracting results only implicit in the literature, and a forthcoming graphical interface that visualizes large-scale structure and permits ad-hoc exploration. (Figure 1 was created by this program.) Future work includes improvements to the graphical subsystem, additional modes to allow the inference system to reason with more complex forms of results, and expansion of the bibliography to encompass more of the field. All of these would make excellent summer projects for undergraduates with appropriate skill sets; in particular, one can read reverse-math papers with few prerequisites, so any interested student could help to expand the Zoo's annotated bibliography while educating themselves on a new active field of mathematical research.

4. Fine Divisions in Reverse Mathematics

Due to the correspondence between RCA_0 and computability, the proofs of reverse-mathematical equivalences typically lend themselves to computability-theoretic analysis. In the last few years, reverse mathematicians (beginning with Dzhafarov) have formalized this to make finer divisions within reverse-mathematical equivalences, by restricting the permissible forms of argument to either computable or uniformly-computable reductions between principles. This was quickly found to be a rediscovery; these uniform reductions were formalized in a different framework in the early 1990's by Weihrauch, for use in computable analysis. As such, Weihrauch reducibility (and its non-uniform variant, computable reducibility) has now become a point of cross-fertilization between the two fields. Under these stricter forms of reduction, we can now separate principles previously considered equivalent, revealing fine structure within reverse mathematics.

I have been particularly interested in the fine structure of weaker principles, and specifically those much weaker than RT^2_2 (Ramsey's theorem for pairs and two colors), which have shown unusual characteristics under analysis by other authors. This began with Hirst's thesis [11], in which he noted that Ramsey's theorem for singletons (RT^1, also known as the infinite pigeonhole principle) was not quite provable in RCA_0 alone, but that adding it to the system did not add much strength; it affects only the first-order consequences, and is equivalent to adding a slightly-strengthened induction principle ($\mathsf{B}\Sigma^0_2$).

Since then, several other such principles have been studied, among them the chain-antichain and ascending-descending-sequence principles:

Definition 4.1 (CAC)**.** Every infinite partial order contains an infinite chain (a total suborder) or antichain (a totally incomparable subset).

Definition 4.2 (ADS)**.** Every infinite linear order contains an infinite increasing sequence or an infinite decreasing sequence.

These follow quickly from applying Ramsey-theoretic reasoning to partial and linear orders respectively, and came to the attention of reverse mathematicians after Herrmann proved that solutions to instances of CAC and RT^2_2 followed the same degree-theoretic pattern; despite this, Hirschfeldt and Shore [10] demonstrated that CAC was strictly weaker than RT^2_2 over RCA_0. They also called attention to its linear-order counterpart ADS, further splitting it from CAC. However, in the process, they constructed a subtle variant on the definition of ADS, proved it to be equivalent to the original over RCA_0, and proceeded to use it for the rest of their reasoning.

On more careful inspection of this last equivalence, the translation is clearly computable, but not uniformly so. In collaboration with Dzhafarov, Solomon, and Suggs [4], I have shown that this is no accident; ADS and this variant (called ADC) are actually two separate principles, distinguishable under Weihrauch reducibility, but separated by only a single non-uniform bit of information — corresponding to whether the sequence found was ascending or descending. Similar analysis further split the "stable" variant of each of these, uncovering six distinct principles where previous work had only shown two. Inspired by this, we found another analysis that applied to partial orders, separating the "stable" part of CAC into two distinct principles WSCAC and SCAC.

Future work. Preliminary results suggest that there is still more structure to be found within weak principles, in particular for principles reverse-mathematically close to Ramsey's theorem for singletons (RT^1); these principles have few (if any) second-order consequences, but prove some first-order statements not themselves provable in RCA_0.

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FIGURE 1. A diagram of some known implications between selected principles in ω-models of RCA_0. All principles depicted are intermediate between ACA_0 and RCA_0.
Generated by the RM Zoo: `http://rmzoo.math.uconn.edu`.

Appendix A. Computability

As mentioned above, we intuitively define a set $A \subseteq \mathbb{N}$ to be *computable* if there is some algorithm that, given $n \in \mathbb{N}$, decides whether $n \in A$. Similarly, a function $f : \mathbb{N} \to \mathbb{N}$ is *computable* if there is an algorithm that, given $n \in \mathbb{N}$, outputs $f(n)$. In fact, we generally state that a set is computable if and only if its characteristic function is computable. A set A is instead said to be *computably enumerable* (c.e.) if given n, an algorithm can confirm if n is in A; if n is not in A, the algorithm need never terminate.

Before continuing, we should note that many sources (including this statement, at certain points) use the term *recursive* rather than computable. This traces to Gödel's formalization of the effectively computable functions as his class of general-recursive functions, and was the standard term of the field for much of its history. However, there has recently been a widespread move within the field, formerly known as "recursion theory," to prefer the term *computability*, motivated by concern for both history and public understanding of the subject.

Gödel, Church, and Kleene all proposed formalizations of computability in the 1930s, but the field as a whole (led prominently by Gödel, who accepted none of these definitions as authoritative, including his own) soon came to agree that the correct definition was that proposed by Turing in his 1936 paper "On Computable Numbers, with an Application to the *Entscheidungsproblem*." [21] Turing's definition is based on simple machines, now called *Turing machines*, with obviously mechanizable operation; in one section of this paper, he proves that these machines are capable of simulating any finite deterministic process executed by any other information-processing machine, and makes the case that even human-implemented algorithms are within the capability of a Turing machine. This claim has since become known as the Church-Turing thesis. Perhaps the most famous result from this paper, though, is Turing's proof that the *halting problem* (the problem of determining whether a Turing machine will ever complete its operation) is not computable.

In his 1939 paper [22], Turing briefly mentions that one could give a Turing machine access to an "oracle", capable of providing solutions to a single given problem, and that if said problem were not itself computable, the resulting *oracle machine* would be able to compute some previously non-computable functions. As developed by Post, starting in 1944 [19], this idea formed the basis of computability theory as we know it today; Post leveraged this idea to formalize the idea of *relative computability* between sets, defining A to be computable relative to B, or *reducible* to B ($A \leq_{\mathrm{T}} B$) if A is computable by a Turing machine given B as an oracle. He also defined stronger notions of reducibility by restricting how the machine could consult the oracle for B. The strongest of these was *1-reducibility*, where $A \leq_1 B$ if there is a computable injection f such that $n \in A$ if and only if $f(n) \in B$.

Each type of *reducibility* induces a preorder on the subsets of $\mathbb{N}$, which in turn induces a partial order on equivalence classes (which we call *degrees*). From Turing reducibility, we obtain the infinite lattice of *Turing degrees*, a central object of study in computability. For our purposes, it is important to note that this lattice exhibits many *minimal pairs*, pairs of degrees $\mathbf{a}$ and $\mathbf{b}$ such that no degree except $\mathbf{0}$ (the computable degree) reduces to both $\mathbf{a}$ and $\mathbf{b}$.

Much of absolute computability generalizes to computability relative to a fixed oracle; we say that these results *relativize*. In particular, the *halting set* (the indices of Turing-machine programs which halt) can be defined relative to a fixed oracle. For a given set A, we call the halting set relative to A the *jump* of A, denoted A'. We always have $A <_{\mathrm{T}} A$; it is easy to show that $A \leq_{\mathrm{T}} A'$, and by Turing's original proof that the halting problem is undecidable, A' cannot be computable relative to A. For instance, the halting set (with no oracle) can be written as $\emptyset'$, but we can also iterate the jump, obtaining the hierarchy $\emptyset <_{\mathrm{T}} \emptyset' <_{\mathrm{T}} \emptyset'' <_{\mathrm{T}} \emptyset''' <_{\mathrm{T}} \cdots$. This hierarchy has proven useful on many occasions, often for its deep connections to sentence complexity; for instance, the sets that are Δ^0_2-definable turn out to be exactly the sets computable in $\emptyset'$.

We will make use of a few other standard computability-theoretic definitions. Using Turing's construction of a universal Turing machine as our model of computation, we are provided with a natural index of Turing machines, with the e-th machine being that simulated by the universal machine with input e. We define the output of the e-th Turing machine, provided some input, to be the e-th *partial computable function*, denoted φ_e; these are only partial functions, as a Turing machine may never halt its computation to produce a final output. A function f is then said to be *diagonally non-computable*, or DNC, if for every e with $\varphi_e(e)$ defined, $f(e) \neq \varphi_e(e)$; that is, if the standard diagonal argument proves f not computable. (Such functions are also referred to as DNR, particularly in topics where the term "recursive" is still preferred over "computable".)

Also, we say that a set A is *high* if $A' \geq_{\mathrm{T}} \emptyset''$. As the following theorem of Martin [18] shows, this class is quite natural: A is high if and only if it computes a function that *dominates* every computable function. That is, A is high if and only if there is some $g \leq_{\mathrm{T}} A$ such that if f is computable, $f(n) \leq g(n)$ for co-finitely many n.

From early in its development, computability has played a large role in the study of what it means for a sequence to be *random*. In fact, modern definitions in the study of *algorithmic randomness* are all strongly tied to ideas of computability. For instance, one of the first attempts at a definition of randomness, by von Mises [23], developed the idea that every reasonable subsequence of a random binary sequence should obey the Law of Large Numbers (i.e., should contain 1's at density-$\frac{1}{2}$). This has since been termed *stochasticity*, since it has generally proven insufficient as a notion of randomness. Church [6] shortly thereafter suggested that the appropriate reasonable notion of selection would be to take computable subsequences; as a result, a sequence that remains unbiased under computable selection is now said to be *Church stochastic*.

Appendix B. Reverse Mathematics

Computability has also surfaced, in the guise of effective construction, in the field of *reverse mathematics*. This area of mathematical logic, founded by Friedman [8], has proven to have a strong cross-disciplinary appeal, drawing the interest of mathematicians of assorted specialties, philosophers of mathematics, and even some cognitive scientists seeking to understand the patterns of information processing. Reverse mathematics is a study of the intuition, common among mathematicians, that certain theorems are "weaker" than, or "essentially the same as," others. However, as long as we work in an axiomatic system in which all of our theorems are true (for instance, second-order arithmetic), implications between theorems are ultimately meaningless tautologies; true statements imply all other true statements, and false statements imply everything.

The insight underlying reverse mathematics is that if we deliberately weaken our base system of axioms and rules, we can speak of the implications between theorems *over* this weak base. All that is left is to choose an appropriate base system, sufficiently weak that it cannot prove too much of mathematics, but strong enough to satisfy our intuition about what can be practically accomplished. Once this is done, we can then say that one theorem is "stronger" than another if assuming the first implies the second over our base system, but the second does not suffice to prove the first.

The field as a whole has almost universally adopted a particular base system, chosen for its success and naturality. In this system, we include almost all of the axioms of Peano arithmetic, though we restrict our notion of induction to Σ^0_1 formulas to avoid accidentally capturing stronger second-order axioms than we intend to include. This leaves us to decide only what explicit set-existence axioms will be included in the system. Since the effectively-describable sets coincide with the computable sets (by the Church-Turing thesis), we choose to assert the existence of sets whose membership is decided by a computable (equivalently, Δ^0_1) formula. This axiom schema is referred to as the Recursive Comprehension Axiom, and thus we call our base system RCA_0.

Early in the development of the field, Friedman, Simpson, and others noticed that the vast majority of theorems successfully analyzed could be proven equivalent to one of the "Big Five" subsystems. These systems are linearly ordered by strength, and each can be taken in turn to correspond to allowing stronger forms of proof. In increasing order of strength, we have:

(1) RCA_0 itself. This corresponds roughly to constructive mathematics, and can prove a somewhat surprising number of standard results, including the Baire category theorem, Urysohn's lemma, and the Tietze extension theorem.
(2) WKL_0, consisting of RCA_0 along with Weak König's Lemma, which states that all infinite binary-branching trees have infinite paths. This can be argued to correspond to proofs reducible to finitistic reasoning, and hence to Hilbert's program of finitistic mathematics. The equivalents of WKL_0 generally match well with pure applications of compactness; for instance, WKL_0 is known equivalent to the separable Hahn-Banach theorem and the Heine-Borel theorem, as well as the existence of a prime ideal in every countable commutative ring.
(3) ACA_0, adding comprehension for all arithmetic formulas (i.e., the existence of sets defined by formulas whose quantifiers range only over $\mathbb{N}$); this has been held to permit all predicative proofs, avoiding ultimately self-referential objects, and is known equivalent to the Bolzano-Weierstrass theorem, Ascoli's theorem, the least upper bound property for $\mathbb{R}$, and the existence of solutions to the halting problem (a foundational problem in computability theory).
(4) ATR_0, permitting transfinite recursion for arithmetic formulas. This system is nearly predicative, but not quite; it is known equivalent to the comparability of countable ordinals, determinacy for open sets in Baire space, and the perfect set theorem (stating that every uncountable closed set in a complete separable metric space contains a perfect closed set).
(5) $\Pi^1_1{-}\mathsf{CA}_0$, adding comprehension for all Π^1_1 formulas (i.e., the existence of sets defined by formulas using only universal quantifiers over sets). This permits most practical impredicative arguments, and is known equivalent to the Cantor-Bendixson theorem and the decomposition of every countable abelian group into a direct sum of a divisible and a reduced group.