THE UNIVERSITY OF CHICAGO

ASYMPTOTIC DENSITY AND EFFECTIVE NEGLIGIBILITY

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY

ERIC P. ASTOR

CHICAGO, ILLINOIS

JUNE 2015

To my fiancée, Erin Martell.

With thanks to my parents.

TABLE OF CONTENTS

LIST OF FIGURES v

ACKNOWLEDGMENTS vi

ABSTRACT viii

1 INTRODUCTION 1
1.1 Context 1
1.2 Standard notation and definitions 2
1.3 Background 3
1.4 Structure and summary 6

2 DENSE COMPUTATION 8
2.1 Introduction 8
2.2 Relations 9
2.3 Upper cones and minimal pairs 13

3 INTRINSIC DENSITY 22
3.1 Introduction 22
3.2 Intrinsic density 25
3.3 Intrinsic density and immunity 28
3.4 Intrinsic density and computability 43
3.5 Intrinsic density in reverse mathematics 48
3.6 Intrinsic density and randomness 55

4 INTRINSIC ASYMPTOTIC COMPUTATION 60
4.1 Introduction 60
4.2 Definitions 61
4.3 Properties of intrinsic asymptotic computation 63

REFERENCES 66

LIST OF FIGURES

2.1 The graph of implications between the notions of asymptotic computability, including computation bounds. 11

3.1 The graphs of the implications between the classical immunity properties 30
3.2 The graph of implications between the classical immunity properties and intrinsic density 0 42

ACKNOWLEDGMENTS

Let me first start by thanking my advisors: Denis Hirschfeldt and Robert Soare have been invaluable to me throughout graduate school, with advice mathematical, professional, and personal alike. The time I have spent with them has led to enormous insights and unusual perspectives, and I appreciate their generosity with their time more than I can say. I would not be a computability theorist if not for my encounters with Bob's enthusiasm and inspiring commitment to the field, and I owe Denis many a debt of gratitude for long discussions and careful debate that shaped both this thesis and my education as a mathematician.

I also owe significant thanks to Carl Jockusch, who so freely shared his time, his insights, and his ideas (among other things, providing the original foundation of asymptotic computability upon which this thesis is built); I thank both Carl and Denis for their gracious consent to include our joint work in Chapter 2 of this thesis.

Others have played a major role in my experience of graduate school, including my academic siblings who shared some time with me at the University of Chicago: Matthew Wright, Jonathan Stephenson, Damir Dzhafarov, and David Diamondstone; I thank them all for their conversations on all topics. In particular, Jonny and I collaborated closely for some time, and I hope to have that opportunity again in the future — hopefully with fewer insurmountable obstructions next time!

Innumerable other logicians have played significant roles in these years of my life, among them Peter Cholak, Rod Downey, Greg Igusa, Maryanthe Malliaris, Timothy McNicholl, and Noah Schweber. I thank them all for their contributions and conversations, not to mention their patience.

Other mathematicians have inspired me along the way as well: Peter May, who helped convince me to spend these years at the University of Chicago; John Boller, Diane Herrmann, and Sarah Ziesler, who have shaped my practice as an academic and teacher; and Paul Schupp, whose collaboration with Carl Jockusch produced the field of asymptotic computability and with whom I had multiple conversations on the further development of these

ideas.

Thanks are also due to those who inspired me in earlier stages of my mathematical career, from Steve Maurer, my undergraduate advisor and mentor in mathematics, to John Boccio, my first undergraduate physics professor; in John's class, I learned how to speak about and collaborate on complex problems, and from Steve, I received my first exposure to the depths of mathematical research. I also thank Richard Piccirillo, my high-school calculus teacher, chess and math-league coach, and advisor, who endorsed every exploration I made beyond the curriculum and into the deeper study of mathematics; his ability in the classroom still remains my model for excellent teaching, despite all the others I have learned from along the way. I would be remiss if I failed to thank Kathy Jarden, my second- and fourth-grade teacher, who encouraged me each time I found a new pattern in mathematics or any other subject we studied; I remember and value the lessons she taught me to this day. All of these teachers helped me discover or stay in touch with the beauty of discovery at the heart of mathematics, and I would not be a mathematician today without them.

I also thank all those I have forgotten to mention; I have never been one with the greatest memory for even the important details of life, and so I am certain I have forgotten to give appropriate credit to far too many people. I beg your forgiveness for my lack of specificity.

However, I have not forgotten my family. From a very young age, my sister, Kate Astor, has always kept me grounded; never discouraging, she still helped me remember humility. To my mother, Stacey Prager: you were never as "bad at math" as you claimed, and your insistence on doing things correctly (even, or perhaps especially, things you disagreed with) has shaped both my life and my mathematics. To my father, Andrew Astor: you played as much of a part in my early mathematical exploration as any of my teachers, and you have been an advisor to me from my earliest days. Thank you all for your encouragement and love, even when I confused you beyond all reasonable understanding.

ABSTRACT

In this thesis, we join the study of asymptotic computability, a project attempting to capture the idea that an algorithm might work correctly in all but a vanishing fraction of cases.

In collaboration with Hirschfeldt and Jockusch, broadening the original investigation of Jockusch and Schupp [12], we introduce *dense computation*, the weakest notion of asymptotic computability (requiring only that the correct answer is produced on a set of density 1), and *effective dense computation*, where every computation halts with either the correct answer or (on a set of density 0) a symbol denoting uncertainty. A few results make more precise the relationship between these notions and work already done with Jockusch and Schupp's original definitions of *coarse* and *generic* computability.

For all four types of asymptotic computation, including generic computation, we demonstrate that non-trivial upper cones have measure 0, building on recent work of Hirschfeldt, Jockusch, Kuyper, and Schupp [9] in which they establish this for coarse computation. Their result transfers to yield a minimal pair for relative coarse computation; we generalize their method and extract a similar result for relative dense computation (and thus for its corresponding reducibility).

However, all of these notions of near-computation treat a set as negligible iff it has asymptotic density 0. Noting that this definition is not computably invariant, this produces some failures of intuition and a break with standard expectations in computability theory. For instance, as shown by Hamkins and Miasnikov [8], the halting problem is (in some formulations) effectively densely computable, even in polynomial time — yet this result appears fragile, as indicated by Rybalov [24].

In independent work, we respond to this by strengthening the approach of Jockusch and Schupp to avoid such phenomena; specifically, we introduce a new notion of intrinsic asymptotic density, invariant under computable permutation, with rich relations to both randomness and classical computability theory. For instance, we prove that the stochasticities corresponding to permutation randomness and injection randomness coincide, and identify

said stochasticity as intrinsic density $\frac{1}{2}$.

We then define sets of intrinsic density 0 to be *effectively negligible*, and classify this as a new immunity property, determining its position in the standard hierarchy from immune to cohesive for both general and Δ^0_2 sets. We further characterize the Turing degrees of effectively negligible sets as those which are either high ($\mathbf{a}' \geq_{\mathrm{T}} \emptyset''$) or compute a DNC (diagonally non-computable) function. In fact, this result holds over $\mathbf{RCA}_0$, demonstrating the reverse-mathematical equivalence of the principles **ID0** and $\mathbf{DOM} \vee \mathbf{DNR}$.

Replacing Jockusch and Schupp's negligibility (density 0) by effective negligibility (intrinsic density 0), we then obtain new notions of intrinsically dense computation. Finally, we generalize Rice's Theorem to all forms of intrinsic dense computation, showing that no set that is 1-equivalent to a non-trivial index set is intrinsically densely computable; in particular, in contrast to ordinary dense computation, we see that the halting problem cannot be intrinsically densely computable.

CHAPTER 1

INTRODUCTION

1.1 Context

For years, there has been strong interest in the distinction between the idealized world of computation and complexity and that of its real-world applications, particularly in problems or algorithms where we find a separation between the worst-case complexity (or, more broadly, difficulty) and the worst cases actually encountered in practice. The simplex algorithm for linear programming is the classic example; there is a family of examples on which the algorithm takes exponential time [16], yet in practice, every problem actually encountered is solved within polynomial time bounds. Even more extreme examples are known, including several problems in group theory (including some variants of the word problem) that are non-computable in general, but for which a low-complexity algorithm solves all examples encountered in practice. [14] In complexity theory, current methods for exploring such structure include the average-case complexity introduced by Gurevich [7] and Levin [17], though this is sensitive to one's choice of probability measure, as well as the smoothed analysis of Spielman and Teng [26]; however, none of these methods have been adapted to computability theory, and it may well be that none are well-suited to such problems.

Taking a more direct approach, several researchers have begun work on the question of whether an algorithm's problematic behavior might be restricted to a negligible set. This is clearly related to the analysts' notion of "almost everywhere", whereby one works modulo sets of measure 0 so as to disregard problematic variations with no practical effect. In a sense, this study is motivated by envy of their methods — in recent years, we have discovered problems that seem to be "computable almost everywhere", and are working to find the right definition for the phrase.

1.2 Standard notation and definitions

In this section, we collect notation and definitions that will be used for the rest of this paper. We will denote the e-th partial computable function (in some fixed universal enumeration) by φ_e.

We routinely identify a set $S \subseteq \omega$ with its characteristic function, $S(n)$, and also with the infinite binary sequence defining its characteristic function, $(S(n))_{n\in\omega}$. By $S\upharpoonright n$, we mean either $S \cap [0, n)$ or the string consisting of the initial n bits of the infinite sequence; which notation we are using at a given moment will be made clear by context. Two sets S and T have *finite difference*, denoted $S =^* T$, if $S(n) = T(n)$ for all sufficiently large n.

Given two finite strings v and w, we say v is a *prefix* of w, denoted $v \preceq w$, if there is a string x such that the concatenation of v followed by x is w (i.e., $vx = w$); this definition extends to infinite sequences w in the natural way. For any string s, we denote the unique prefix of s of length n by $s\upharpoonright n$.

The prefix-free Kolmogorov complexity of a binary string s is denoted as $K(s)$; we refer to Downey and Hirschfeldt [3] or Nies [21] for the details of its definition and properties, but note that it does relativize: we can consider the prefix-free Kolmogorov complexity of s with respect to A, denoted $K^A(s)$. Both of these books also provide many equivalent characterizations of a 1-random set; for this paper, we will take the characterization in terms of the Kolmogorov complexity of initial segments as our definition. A set $S \subseteq \omega$ is *1-random* if there is some constant c such that $K(S\upharpoonright n) \geq n - c$ for all n. This definition inherits a natural relativization from prefix-free complexity: S is 1-random relative to A if there is some c such that $K^A(S\upharpoonright n) \geq n - c$ for all n. In general, S is said to be *n-random* if S is 1-random relative to $\emptyset^{(n-1)}$.

There is an equivalent characterization of randomness in terms of tests of limited descriptive complexity; we will not discuss it in detail. However, there is a natural weakening of n-randomness that comes from this characterization: a set $S \subseteq \omega$ is said to be *weakly n-random* if it belongs to no Π^0_n class of measure 0.

A set $S \subseteq \omega$ is *1-generic* if, for every c.e. set X of finite binary strings, there is some initial segment $\sigma \prec S$ such that either $\sigma \in X$ or $\sigma \not\preceq \tau$ for every $\tau \in X$.

We say that a function f is *dominant* if for all e such that φ_e is total, $f(n) \geq \varphi_e(n)$ for all sufficiently large n.

If a function f has $f(e) \neq \varphi_e(e)$ whenever $\varphi_e(e) \downarrow$, it satisfies the classic diagonal argument that proves it not on the list of computable functions; we say that f is *diagonally non-computable*, or DNC. A set A is said to have DNC degree if there is an A-computable DNC function.

1.3 Background

The essential difficulty in defining "computable almost everywhere" is that there is no uniform probability measure on the integers, and thus no natural notion of a null set. Instead, if we want a uniform measurement of the size of a subset of ω, we are forced to abandon countable additivity and fall back to pseudo-measures. One of the most practical is asymptotic density.

Definition 1.3.1. Let $S \subseteq \omega$, where $\omega = \{0, 1, 2, \ldots\}$ is the set of natural numbers. For every $n \geq 0$, we denote $S \cap [0, n)$ by $S \upharpoonright n$.

We define the n-th *partial density* of S as

$$\rho_n(S) \coloneqq \frac{|S \upharpoonright n|}{n}.$$

The *lower density* $\underline{\rho}(S)$ of S is

$$\underline{\rho}(S) \coloneqq \liminf_{n \to \infty} \rho_n(S) = \liminf_{n \to \infty} \frac{|S \upharpoonright n|}{n},$$

and the *upper density* $\overline{\rho}(S)$ of S is

$$\overline{\rho}(S) \coloneqq \limsup_{n\to\infty} \rho_n(S) = \limsup_{n\to\infty} \frac{|S \upharpoonright n|}{n}.$$

If the limit of the partial densities exists (i.e., $\underline{\rho}(S) = \overline{\rho}(S)$), then we say that S has *(asymptotic) density*

$$\rho(S) \coloneqq \lim_{n\to\infty} \rho_n(S) = \lim_{n\to\infty} \frac{|S \upharpoonright n|}{n}.$$

Of course, $0 \leq \underline{\rho}(S) \leq \overline{\rho}(S) \leq 1$ for all $S \subseteq \omega$. In an unfortunate collision of terms, at least for computability-theoretic work, a set is said to be *generic* if it has density 1 (equivalently, $\underline{\rho}(S) = 1$). The name is motivated by the fact that given a generic set S, the probability that a random integer selected from $[0, n)$ will lie in S approaches 1 as n increases; thus, in some sense, such a set contains all generic integers. The complement of this notion is more useful for our purposes:

Definition 1.3.2. A set $S \subset \omega$ is said to be *negligible* if it has density 0 (equivalently, if $\overline{\rho}(S) = 0$).

Once we fix a notion of negligibility in ω, the problem of defining computability modulo a negligible set is reduced to determining the acceptable forms of error. In 2003, Kapovich, Myasnikov, Schupp, and Shpilrain [14] introduced generic-case complexity, considering algorithms that committed no error, but could fail to converge or otherwise violate a complexity bound on a set of density 0. They showed that this captured the phenomenon observed in several group-theoretic problems with non-computable instances that in practice have been simple to solve; for instance, they demonstrated that for any G in an extremely large class of groups, the word problem for G has linear-time generic-case complexity. Myasnikov, in collaboration with Hamkins, went on to apply these ideas to Turing's halting problem [8], and proved that (for reasons having to do with the prevalence of trivially halting or trivially non-halting programs in many models of computation) the halting problem is "generic-case decidable" in said models. This was later refined by Rybalov [24], who proved that the

halting problem is *not* "strongly generic-case decidable" (that is, decidable modulo sets with partial density converging to 0 exponentially fast); this proof, by contrast, is valid for all Turing-machine models of computation.

Jockusch and Schupp [12] have since defined and begun the study of the computability theory corresponding to generic-case complexity, and more generally the relations between asymptotic density and computability. Their work has been further developed in collaboration with Downey [5] and McNicholl [6], and refined in specific cases by Igusa [10] and Bienvenu, Day, and Hölzl [1].

Following Kapovich, Myasnikov, Schupp, and Shpilrain [14] in permitting no false convergence, Jockusch and Schupp [12] defined one form of asymptotic computability as follows:

Definition 1.3.3. A partial function $f : \omega \to \omega$ is a *partial description* of a function $g : \omega \to \omega$ if $f(n) = g(n)$ whenever $f(n)$ converges. If f has density-1 domain, we call f a *generic-case description* of g.

We say that $g : \omega \to \omega$ is *computable in the generic case*, or *generic-case computable*, if g has a computable generic-case description.

When specialized to sets, Jockusch and Schupp [12] noted an equivalent formulation that provided some insight and has proved extremely useful.

Proposition 1.3.4. *A set $A \subseteq \omega$ is generic-case computable iff it is densely approximable by c.e. sets; that is, iff there exist c.e. sets A_0 and A_1 such that $A_1 \subseteq A$, $A_0 \subseteq \overline{A}$, and $A_0 \sqcup A_1$ has density 1.*

By analogy, they also suggested (and analyzed) another variant; in coarse computation, we permit only errors of commission, requiring that descriptions be total while allowing convergence to an incorrect answer on a negligible set. [12]

Definition 1.3.5. A total function $f : \omega \to \omega$ is a *coarse description* of a function $g : \omega \to \omega$ if $f(n) = g(n)$ on a set of density 1.

We say that $g : \omega \to \omega$ is *coarsely computable* if it has a computable coarse description.

Again, specialized to sets, this yields a useful and equivalent formulation:

Proposition 1.3.6. *A set $A \subseteq \omega$ is coarsely computable iff there is a computable set C such that $A \triangle C$ has density 0. [12]*

Downey, Jockusch, and Schupp [5] then generalized each of these definitions, noting that we can of course relax the condition that convergence (or correctness) is achieved on a set of density 1 to a set of density r.

Definition 1.3.7. We say that a function g is *(asymptotically) computable at density r* if it has a computable partial description with domain having lower density at least r. We then define the *(asymptotic) computation bound* for g to be

$$\alpha(g) = \sup\{r : g \text{ is computable at density } r\}.$$

Similarly, if there is a total computable function f such that $f(n) = g(n)$ on a set of lower density at least r, we say that g is *coarsely computable at density r*. We thus obtain the *coarse computation bound* for g, defined as $\gamma(g) = \sup\{r : g \text{ is coarsely computable at density } r\}$.

1.4 Structure and summary

In the first chapter of this thesis, Chapter 2, we begin by generalizing the definitions of Jockusch and Schupp [12] and Downey, Jockusch, and Schupp [5] to allow us to discuss asymptotic computability more broadly. In every variant of asymptotic computability, we find that any function that is asymptotically computable below a positive-measure subset of 2^ω is in fact asymptotically computable and show as a consequence that minimal pairs exist for relative dense computation, in parallel to an argument of Hirschfeldt, Jockusch, Kuyper, and Schupp [9] establishing this for relative coarse computation.

Next, we revert to a direct discussion of what it should mean for a set to be *effectively* negligible. In Chapter 3, we discuss the difficulties that result from using asymptotic density

alone to define negligibility, and propose a more computability-theoretic definition of *intrinsic density* by insisting on invariance under computable permutations. Studying the resulting notion of *effective negligibility*, we find that it is a new notion of immunity in the sense of Post's program, and characterize its precise relation to this classical hierarchy. We also study the Turing degrees of effectively negligible sets, discovering that all such sets have particular types of inherent computational power. As a result, we make a reverse-mathematical discovery, relating the existence of effectively negligible sets to the disjunction of two more-standard principles. Lastly, we note that taking an intermediate value of intrinsic density reproduces a form of stochasticity implicit in Miller and Nies [19]; this highlights connections between stochasticity and immunity, providing another connection between algorithmic randomness and classical computability theory.

Finally, in Chapter 4, we apply our effective negligibility to produce new definitions of *intrinsic asymptotic computation*, allowing us to discuss whether functions (or sets) are inherently computable "almost everywhere." We show that such notions are non-trivial weakenings of standard Turing computability; in particular, to demonstrate that there are sets that are not intrinsically computable, we generalize Rice's Theorem to show that non-trivial index sets cannot be intrinsically computable. As a corollary, we see that (by contrast to ordinary asymptotic computation), the halting problem is not intrinsically computable.

CHAPTER 2

DENSE COMPUTATION

This chapter is developed on the basis of joint work with Denis Hirschfeldt and Carl Jockusch.

2.1 Introduction

Though Jockusch and Schupp [12] defined forms of asymptotic computation permitting errors of omission (generic-case computation) and errors of commission (coarse computation), obvious variants remain to be analyzed. In particular, we may define forms of asymptotic computation in which both forms of error may be present, or require that our computations emit a signal whenever error may be possible.

The former of these is the weakest form of asymptotic computation. We place no requirements on how the description may fail to agree with the function described, and require only that these failures are restricted to a negligible set of inputs:

Definition 2.1.1. A partial function $f : \omega \to \omega$ is a *dense description* of a function $g : \omega \to \omega$ if $f(n) \downarrow = g(n)$ on a set of density 1.

We say that $g : \omega \to \omega$ is *densely computable* if it has a computable dense description.

Of course, any function that is either coarsely or generic-case computable must be densely computable, as all generic-case or coarse descriptions are dense descriptions.

At the other extreme, we require descriptions that must either answer correctly or halt with a signal that they are not able to give the required answer:

Definition 2.1.2. A total function $f : \omega \to (\{-1\} \sqcup \omega)$ is an *effective description* of a function $g : \omega \to \omega$ if for all n, $f(n) = g(n)$ or $f(n) = -1$. For such an f, its *effective domain* is defined to be $f^{-1}(\omega)$. If f has effective domain of density 1, we say that f is an *effective dense description*.

We say that $g : \omega \to \omega$ is *effectively densely computable* if g has a computable effective dense description.

In each case, we can generalize to sets of lower density and, as for coarse and generic-case computation, define corresponding computability bounds.

Definition 2.1.3. We say that a function g is *effectively computable at density* r if g has an effective description with effective domain of lower density at least r, and define the *effective computation bound* of g to be $\beta(g) = \sup\{r : g \text{ is effectively computable at density } r\}$.

Similarly, if there is a p.c. function f such that $f(n) \downarrow = g(n)$ on a set of lower density at least r, we say that g is *partially computable at density* r. We thus obtain the *partial computation bound* for g, defined as $\delta(g) = \sup\{r : g \text{ is partially computable at density } r\}$.

In Section 2.2, we begin our discussion by establishing the relations between all four notions of asymptotic computability, including their corresponding computation bounds. These prove to be relatively straightforward; our results are summarized in Figure 2.1.

In Section 2.3, we then move to a question of "degree structure", or classical computability; specifically, we ask whether each notion of relative asymptotic computability might admit minimal pairs, as exist in the Turing degrees. We make partial progress in all cases, and demonstrate that in fact there are minimal pairs for relative *dense* computability.

2.2 Relations

When considering these new notions, our first order of business is to establish the relations between all four notions of asymptotic computability, including their corresponding computation bounds. Jockusch and Schupp [12] have already established that there are sets which are coarsely computable but not generic-case computable, and vice-versa, and so that generic-case and coarse computability are incomparable. We begin by establishing some relations between effective dense computation and the other three notions.

Proposition 2.2.1. *The following are equivalent:*

1. $g : \omega \to \omega$ *is effectively densely computable.*

2. *g has a computable generic-case description f with computable domain.*

3. *g has a computable coarse description f such that $f(n) = g(n)$ on a computable set of density 1.*

4. *g has a computable dense description f such that $f(n) \downarrow= g(n)$ on a computable set of density 1.*

Proof. (1) $\implies$ (2) $\wedge$ (3): Let f be a computable effective dense description of g. Since f is computable, $I = f^{-1}(-1)$ is computable. We thus define the partial computable function $\widehat{f} : \omega \to \omega$ by

$$\widehat{f}(n) = \begin{cases} f(n) & n \notin I, \\ \uparrow & n \in I, \end{cases}$$

and the total computable function $f_0 : \omega \to \omega$ by

$$f_0(n) = \begin{cases} f(n) & n \notin I, \\ 0 & n \in I. \end{cases}$$

Since I has density 0, $\widehat{f}$ has density-1 domain and agrees with f (which agrees with g) on its entire domain. Therefore, $\widehat{f}$ is a computable generic-case description of g with density-1 domain.

Similarly, f_0 agrees with f (which agrees with g) on $C = \omega \setminus I$, so f_0 is a computable coarse description of g that agrees with g on a computable set of density 1.

(2) $\vee$ (3) $\implies$ (4): Trivial.

(4) $\implies$ (1): Let f be a computable dense description of g such that, for some computable set C of density 1, $f(n) \downarrow= g(n)$ for all $n \in C$. We define $\widehat{f} : \omega \to (\{-1\} \sqcup \omega)$ by

$$\widehat{f}(n) = \begin{cases} f(n) & n \in C, \\ -1 & n \notin C. \end{cases}$$

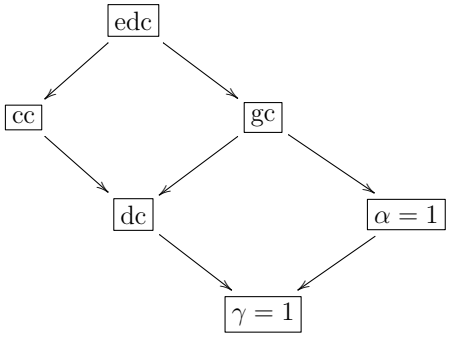


Figure 2.1: The graph of implications between the notions of asymptotic computability, including computation bounds. All implications shown are strict, and all not shown are false. We have abbreviated coarse computability by "cc", generic-case computability by "gc", and (effective) dense computability by "(e)dc".

Since $f(n) \downarrow= g(n)$ for all $n \in C$, $\widehat{f}$ is total and, for all n, $\widehat{f}(n) = g(n)$ or $\widehat{f}(n) = -1$. Therefore, $\widehat{f}$ is a computable effective dense description of g. □

In particular, we therefore see that any set that is effectively densely computable is also both coarsely computable and generic-case computable. However, the converse is easily seen to be false; any density-1 c.e. set with no density-1 computable subset (as constructed by Jockusch and Schupp [12]) is both coarsely and generic-case computable, but can have no generic-case description with computable domain. Therefore, effective dense computability is strictly stronger than the conjunction of coarse and generic-case computability.

Theorem 2.2.2. *For all $g : \omega \to \omega$, $\alpha(g) = \beta(g)$ and $\gamma(g) = \delta(g)$; that is, the computation and effective computation bounds always coincide, as do the coarse computation and partial computation bounds.*

Proof. Suppose g is partially computable at density d; that is, there is a partial computable function f such that $\underline{\rho}(\{n : f(n) \downarrow= g(n)\}) \geq d$. We denote $\{n : f(n) \downarrow= g(n)\}$ by S, and fix some $\varepsilon > 0$.

Let D be the domain of f; by Theorem 3.9 of Downey, Jockusch, and Schupp [5], since

D is c.e., it has a computable subset B with $\underline{\rho}(B) > \underline{\rho}(D) - \varepsilon$. We define $\widehat{f}$ by

$$\widehat{f}(n) = \begin{cases} f(n) & n \in B, \\ 0 & n \notin B. \end{cases}$$

This is clearly a total computable function. Since $S \subseteq D$ has lower density at least d, it must be that $S \cap B$ has lower density at least $d - \varepsilon$; however, $S \cap B$ is precisely the set of positions at which $\widehat{f}(n) = g(n)$. This shows that A is coarsely computable at density $d - \varepsilon$ for all $\varepsilon > 0$.

Therefore, we see that $\gamma(g) \geq \delta(g)$. Since any coarse computation at density d is also a partial computation at density d, we in fact have that $\gamma(g) = \delta(g)$.

A similar argument shows that $\alpha(g) = \beta(g)$. □

Corollary 2.2.3. *Every densely computable set has $\gamma = 1$.*

To complete our diagram of implications (Figure 2.1), it only remains to separate $\alpha = 1$ from coarse and dense computability. First, we note that a modification of the simple set construction (discussed in Jockusch and Schupp [12]) yields a simple set with density 0. Any such set is coarsely computable, but has $\alpha = 0$; therefore, there are coarsely computable sets without $\alpha = 1$.

Resolving the other non-implication is also straightforward, but takes slightly more work.

Proposition 2.2.4. *There is a set with $\alpha = 1$ that is not densely computable.*

Proof Sketch. $R(A) = \bigcup_{n \in A} \{2^n k : k \text{ is odd}\}$ is such a set for any non-Δ^0_2 set A. Knowledge of $A \upharpoonright n$ suffices to effectively densely compute $R(A)$ at density $1 - 2^{-n}$, so $\alpha(R(A)) = 1$. However, this set cannot be densely computable.

Suppose $R(A)$ were densely computable, and let φ be a computable dense description of $R(A)$ such that, for each s, there is at most one n_s such that $\varphi(n)[s] \downarrow$ and $\varphi(n)[s-1] \uparrow$. Let $\widehat{f}(n, s)$ be the plurality-vote value of $\{\varphi(2^n k)[s] : [k < s] \wedge [\varphi(2^n k)[s] \text{ converges}]\}$, and define $f(n) = \lim_{s \to \infty} \widehat{f}(n, s)$.

If $\widehat{f}(n,s)$ were to disagree with $A(n)$ for infinitely many s, φ could not be a dense description of A. Therefore, there must be some t such that $\widehat{f}(n,s) = A(n)$ for all $s > t$, so $f(n) = A(n)$ for all n. However, f is computable in $\widehat{f}' = \varphi' = \emptyset'$.

Therefore, if $R(A)$ is densely computable, A must be Δ^0_2. □

There are certain questions suggested by Figure 2.1 that remain open. As noted above, effective dense computability is strictly stronger than the conjunction of coarse and generic-case computability. However, three similar questions remain, which we suggest as directions for future research in the field; the required constructions to answer these questions seem likely to provide some useful insight into these notions of asymptotic computation.

Open Question 2.2.1. Is there a densely computable set A with $\alpha(A) = 1$ that is not generic-case computable? That is, is generic-case computability strictly stronger than the conjunction of dense computability with computability bound $\alpha = 1$?

Open Question 2.2.2. Is there a densely computable set that is neither coarsely nor generic-case computable? That is, is dense computability simply the disjunction of coarse and generic-case computability?

Open Question 2.2.3. Is there a set A with $\gamma(A) = 1$ that has is not densely computable $\alpha(A) < 1$ and is not densely computable? That is, is $\gamma(A) = 1$ iff A is densely computable or $\alpha(A) = 1$?

2.3 Upper cones and minimal pairs

Now that we have resolved the relations between the various notions of asymptotic computation, we shift to a question of their corresponding degree structure. Specifically, we ask whether minimal pairs exist for each notion of relative asymptotic computation, wherein we say that A is (effectively) densely, coarsely, or generic-case computable *relative to* B if there exists a B-computable appropriate asymptotic description. This inquiry was launched by

Downey, Jockusch, and Schupp [5] for generic-case computation, and partially resolved in that case by Igusa [10], who showed that no minimal pair existed for relative generic-case computation.

However, this was only a partial resolution; no relative asymptotic computability is transitive, so they do not immediately give a degree structure. After all, even if A computes φ_0, an effective dense description of B, and B computes φ_1, an effective dense description of C, there need not be a way to compute φ_1 from φ_0! Instead, our notions of reducibility must be reductions to asymptotic descriptions; such reducibilities are defined in Jockusch and Schupp [12] via enumeration operators, though we can also work more directly with the concept of reduction to asymptotic descriptions. For each notion of asymptotic computation, the corresponding asymptotic reduction is stronger than relative asymptotic computability. Thus, for a full proof that an asymptotic computability has no minimal pairs, one must show that the corresponding reducibility has no minimal pairs.

By contrast, for a positive proof of the existence of minimal pairs, it suffices to construct a minimal pair for the relative asymptotic computability. As our results herein are all positive, we will not discuss any asymptotic reducibilities in detail.

Hirschfeldt, Jockusch, Kuyper, and Schupp [9] proved the first positive result in this project, showing that all non-trivial upper cones for relative coarse computation have measure 0, and transferred this result to demonstrate the existence of many minimal pairs for coarse computation (and hence for coarse reducibility). In this section, we generalize their argument, demonstrating that in fact non-trivial upper cones for all notions of relative asymptotic computability have measure 0, and in the case of dense computation, also extend their transfer argument to construct minimal pairs for dense reducibility with slightly higher complexity.

We begin with a technical lemma, in essence a version of Fubini's theorem allowing us to relate asymptotic density and Lebesgue measure on Cantor space.

Lemma 2.3.1. *If $(1-q)a+b>1$, there is no sequence $\{\mathcal{S}_n \subseteq 2^\omega\}_{n\in\omega}$ such that*

$$\overline{\rho}(\{n : \mu(\mathcal{S}_n) < q\}) > a$$

and, taking $S(A) = \{n : A \in \mathcal{S}_n\}$,

$$\mu(\{A \in 2^\omega : \rho(S(A)) = 1\}) > b.$$

Proof. Suppose such a sequence exists. Fixing some $r < 1$, consider the classes

$$\mathcal{X}_n = \big\{A \in 2^\omega : (\forall p > n)\rho_p(S(A)) > r\big\}.$$

Since the union of these classes includes $\{A \in 2^\omega : \rho(S(A)) = 1\}$, it must have measure $> b$; since the classes are nested, there must be some finite N such that $\mu(\mathcal{X}_n) > b$ for all $n > N$.

As the $\mathcal{S}_j$ with measure $< q$ have upper density greater than a, there must be some $n > N$ such that $\mu(\mathcal{S}_j) < q$ for at least an values of $j \in [1, n]$.

As integration commutes with finite sums,

$$\frac{1}{n}\sum_{j=1}^{n}\int_{2^\omega} \mathbb{1}_{\mathcal{S}_j}\, d\mu = \int_{2^\omega} \frac{1}{n}\sum_{j=1}^{n} \mathbb{1}_{\mathcal{S}_j}\, d\mu.$$

However,

$$\frac{1}{n}\sum_{j=1}^{n}\int_{2^\omega} \mathbb{1}_{\mathcal{S}_j}\, d\mu = \frac{1}{n}\sum_{j=0}^{n-1}\mu(\mathcal{S}_j) < 1-(1-q)a$$

but

$$\int_{A\in 2^\omega} \frac{1}{n}\sum_{j=1}^{n} \mathbb{1}_{\mathcal{S}_j}\, d\mu = \int_{A\in 2^\omega} \rho_n(S(A))\, d\mu > \int_{\mathcal{X}_n} \rho_n(S(A))\, d\mu > \mu(\mathcal{X}_n)r > br.$$

Therefore, we must have $1-(1-q)a > br$; rearranging, we see that

$$(1-q)a + br < 1$$

for all $r < 1$. Therefore, we conclude that $(1-q)a + b \leq 1$. □

Putting this into a more useful form for our purposes, we arrive at the following theorem:

Theorem 2.3.2. *For any sequence* $\{\mathcal{S}_n \subseteq 2^\omega\}_{n\in\omega}$*, if*

$$\mu(\{A \in 2^\omega : \rho(S(A)) = 1\}) > q, \tag{*}$$

we must have $\rho(\{n : \mu(\mathcal{S}_n) \geq q\}) = 1$.

Proof. Suppose we have such a sequence. Since $\mu(\{A \in 2^\omega : \rho(S(A)) = 1\}) > q$, there is some $\varepsilon > 0$ such that

$$\mu(\{A \in 2^\omega : \rho(S(A)) = 1\}) > q + \varepsilon.$$

Let $p = \overline{\rho}(\{n : \mu(\mathcal{S}_n) < q\})$ and assume, towards a contradiction, that $p > 0$. We then define $I = \{n_j\}$ to be the sequence of n such that $\mu(\mathcal{S}_n) < q$ or $n \equiv 0 \pmod{\lceil \frac{1}{p\varepsilon} \rceil}$. This sequence has lower density at least $p\varepsilon$, and upper density at most $p(1+\varepsilon)$. We then pass to the subsequence $\left\{\mathcal{S}_{n_j}\right\}$. Since I has positive lower density, every density-1 set must intersect I with density 1 within I; therefore, property (*) is preserved. Since the upper density of $\{n_j\}$ is at most $p(1+\varepsilon)$, but $\{n_j\}$ contains all n such that $\mu(\mathcal{S}_n) < q$ (which have upper density p), we must have that

$$\overline{\rho}(\{j : \mu(\mathcal{S}_{n_j}) < q\}) \geq \frac{1}{1+\varepsilon} > 1 - \varepsilon.$$

Thus, we have constructed a sequence having $\overline{\rho}(\{n : \mu(\mathcal{S}_n) < q\}) > 1 - \varepsilon$ and satisfying property (*).

However, this sequence meets the conditions for Lemma 2.3.1, taking $a = 1 - \varepsilon$ and

$b = q + \varepsilon$. Since $(1-q)a + b = (1-q)(1-\varepsilon) + q + \varepsilon = 1 + q\varepsilon > 1$, this would actually contradict the lemma, so we must have $p = 0$. Therefore, $\rho(\{n : \mu(\mathcal{S}_n) \geq q\}) = 1$. □

This theorem forms the foundation of our arguments in the remainder of this Section, allowing us to use majority-vote arguments to construct computable asymptotic descriptions for any set with a positive-measure upper cone. We first apply this to generic-case computation, obtaining a result that provides a useful contrast with that of Igusa [10].

Theorem 2.3.3. *If A is not generic-case computable, then*

$$\mu(\{X \in 2^\omega : A \text{ is generic-case computable relative to } X\} = 0,$$

and in particular, no element of this set can be weakly 4-random relative to A.

Proof. By countable additivity, it suffices to prove that $\mu(\mathcal{A}_\Phi) = 0$, where

$$\mathcal{A}_\Phi = \{X \in 2^\omega : \Phi^X \text{ is a generic-case description of } A\}.$$

for a fixed computable 0-1-valued functional Φ.

Towards a contradiction, suppose otherwise. By Lebesgue density, we can assume that $\mu(\mathcal{A}_\Phi) > \frac{3}{4}$. Define $\mathcal{S}_n = \left\{X \in 2^\omega : \Phi^X(n) \downarrow = A(n)\right\}$. By the theorem, since $\mu(\{X \in 2^\omega : \rho(S(X)) = 1\}) \geq \mu(\mathcal{A}_\Phi) > \frac{3}{4}$, we must have $\mu(\mathcal{S}_n) \geq \frac{3}{4}$ for all but a density-0 set of n's. We can use this to compute a generic-case description of A, using a majority-rule scheme.

Specifically, to calculate $A(n)$, we wait to see $\Phi^X(n)$ converge on a class with measure at least $\frac{2}{3}$, and let $f(n)$ be the majority-rule value (for concreteness, breaking ties in favor of 0). f is, by the use principle, a partial computable function.

Note that $f(n)$ cannot converge to a different value than $A(n)$; otherwise, we have a class $\mathcal{I}_n$ of measure at least $\frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$ for which $\Phi^X(n) \downarrow$ and $\Phi^X(n) \neq A(n)$. Since no $X \in \mathcal{I}_n$ is a generic-case description of A, it must be that $\mathcal{I}_n$ and $\mathcal{A}_\Phi$ are disjoint; however, $\mu(\mathcal{A}_\Phi) > \frac{3}{4}$ and $\mu(\mathcal{I}_n) \geq \frac{1}{3}$, so this is impossible.

Therefore, the constructed function is a computable partial description of A, with $f(n) = A(n)$ whenever $f(n)\downarrow$. It only remains to show that f converges on a set of density 1. However, by construction, $f(n)$ must converge whenever $\mu(\mathcal{S}_n) \geq \frac{2}{3}$. Since $\rho(\{n : \mu(\mathcal{S}_n) \geq \frac{3}{4}\}) = 1$, f must converge on a set of density 1. Therefore, A is generic-case computable, in contradiction to our assumptions.

Writing out the definition of $\mathcal{A}_\Phi$ carefully, we see that $\mathcal{A}_\Phi$ is a Π_4^A-class; since $\mu(\mathcal{A}_\Phi) = 0$, $\mathcal{A}_\Phi$ contains no set that is weakly 4-random relative to A. □

Obviously, as Igusa's result shows that no minimal pairs exist for relative generic-case computation, any attempt to use the method of Hirschfeldt, Jockusch, Kuyper, and Schupp [9] to extract minimal pairs from this theorem must fail. This will be discussed at greater length in a published paper to follow.

A similar proof gives us the same result for dense computability, though it does require slightly more attention to control the possibility of error in the computations monitored by f.

Theorem 2.3.4. *If A is not densely computable, then*

$$\mu(\{X \in 2^\omega : A \text{ is densely computable relative to } X\} = 0,$$

and in particular, no element of this set can be weakly 4-random relative to A.

Proof. By countable additivity, it suffices to prove that $\mu(\mathcal{A}_\Phi) = 0$, where

$$\mathcal{A}_\Phi = \{X \in 2^\omega : \Phi^X \text{ is a dense description of } A\}.$$

for a fixed computable 0-1-valued functional Φ.

Towards a contradiction, suppose otherwise. By Lebesgue density, we can assume that $\mu(\mathcal{A}_\Phi) > \frac{3}{4}$. Define $\mathcal{S}_n = \left\{X \in 2^\omega : \Phi^X(n)\downarrow = A(n)\right\}$. By the theorem, we must have

$\mu(\mathcal{S}_n) \geq \frac{3}{4}$ for all but a density-0 set of n's. We can use this to compute a dense description of A, using a majority-rule scheme.

Specifically, to calculate $A(n)$, we wait to see $\Phi^X(n)$ converge on a class with measure at least $\frac{2}{3}$. If this happens, we declare $f(n)$ to be the majority value (again, breaking ties in favor of 0). f is, by the use principle, a partial computable function. It only remains to show that f converges correctly to A on a set of density 1.

By the definition of $\mathcal{S}_n$, we know that $\Phi^X(n)$ converges for every $X \in \mathcal{S}_n$. Since if $f(n)$ diverges, $\Phi^X(n)$ fails to converge on a class of measure at least $\frac{2}{3}$, we must have $\mu(\mathcal{S}_n) < \frac{2}{3}$ in this case. If instead $f(n)$ converges incorrectly, we have a set of measure at least $\frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$ on which $\Phi^X(n) \downarrow = 1 - A(n)$; therefore, $\mu(\mathcal{S}_n) \leq \frac{2}{3}$.

Since $\mu(\mathcal{S}_n) \geq \frac{3}{4}$ for all but a density-0 set of n's, each case can only occur on a set of density 0. We conclude that $f(n)$ must converge to $A(n)$ on a set of density 1. Therefore, f is a dense description of A, and so A is densely computable, in contradiction to our assumptions.

Again, writing out the definition of $\mathcal{A}_\Phi$ carefully, we see that $\mathcal{A}_\Phi$ is a Π^A_4-class; since $\mu(\mathcal{A}_\Phi) = 0$, $\mathcal{A}_\Phi$ contains no set that is weakly 4-random relative to A. □

In this case, a slight variant on the method of Hirschfeldt, Jockusch, Kuyper, and Schupp [9] will allow us to use randomness to generate minimal pairs for dense reducibility. Their transfer argument is based on the observation that any two coarse descriptions of the same set are necessarily coarse descriptions of each other. The analogous statement makes little sense for dense descriptions, as dense descriptions are partial functions; however, by careful choice of a completion of one description, we can recover their construction of minimal pairs with little added complexity.

Corollary 2.3.5. *If Y is not densely computable and X is weakly 4-random relative to Y, then X and Y are a minimal pair for relative dense computability (and hence for dense reducibility).*

Proof. Suppose C is densely computable relative to Y and relative to X. That is, suppose there is some Φ such that Φ^X and Φ^Y are both dense computations of C.

Let P be some set both PA and low over Y. Since $P \gg Y$, P computes a $\{0,1\}$-valued completion of Φ^Y; let $D \leq_{\mathrm{T}} P$ be such a completion. Since Φ^Y converges and agrees with C on a density-1 set, we note that D is in fact a coarse description of C. In particular, since Φ^X is a dense computation of C, it must also be a dense computation of D.

However, since $D \leq_{\mathrm{T}} P$ and P is low over Y, X is also weakly 4-random relative to D. Since D is densely computable relative to X, by Theorem 2.3.4, we conclude that D must be densely computable. Since any dense computation of D is a dense computation of C, C must also be densely computable. Therefore, X and Y are a minimal pair for relative dense computability. □

Our argument goes through with only minor changes for effective dense computation; combining our work for generic-case and dense computation, we can ensure that all convergence is correct while also signaling in finite time whether or not convergence will occur.

Theorem 2.3.6. *If A is not effectively densely computable, then*

$$\mu(\{X \in 2^\omega : A \text{ is effectively densely computable relative to } X\}) = 0,$$

and in particular, no element of this set can be weakly 4-random relative to A.

Proof. By countable additivity, it suffices to prove that $\mu(\mathcal{A}_\Phi) = 0$, where

$$\mathcal{A}_\Phi = \{X \in 2^\omega : \Phi^X \text{ is an effective dense description of } A\}$$

for a fixed computable functional Φ. Specifically, we use the definition whereby Φ^X is an effective dense description of A iff it is a total function with values from $\{0,1,2\}$, where $(\Phi^X)^{-1}(\{2\})$ has density 0 and $\Phi^X(n) \neq 1 - A(n)$ for all n.

Towards a contradiction, suppose otherwise. By Lebesgue density, we can assume that

$\mu(\mathcal{A}_\Phi) > \frac{7}{8}$. Define $\mathcal{S}_n = \left\{X \in 2^\omega : \Phi^X(n) \downarrow = A(n)\right\}$. By the theorem, we must have $\mu(\mathcal{S}_n) \geq \frac{7}{8}$ for all but a density-0 set of n's. We can use this to compute an effective dense description of A, using a majority-rule scheme.

Specifically, to calculate $A(n)$, we wait to see $\Phi^X(n)$ converge on a class with measure at least $\frac{3}{4}$, and let $f(n)$ be the plurality-rule value (for concreteness, breaking ties in favor of 2, then 0). f is, by the use principle, a partial computable function. Since $\mu(\mathcal{A}_\Phi) > \frac{7}{8} > \frac{3}{4}$, and every effective dense description is total, f must also be total.

Note that $f(n)$ cannot converge to $1 - A(n)$; otherwise, we have a class $\mathcal{I}_n$ of measure at least $\frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$ on which $\Phi^X(n) \downarrow = 1 - A(n)$. Since for no $X \in \mathcal{I}_n$ is Φ^X an effective dense description of A, it must be that $\mathcal{I}_n$ and $\mathcal{A}_\Phi$ are disjoint; however, $\mu(\mathcal{A}_\Phi) > \frac{7}{8}$ and $\mu(\mathcal{I}_n) \geq \frac{1}{4}$, so this is impossible.

If $f(n)$ converges to 2, we have a class of measure at least $\frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$ on which $\Phi^X(n)$ converges to 2; therefore, $\mu(\mathcal{S}_n) \leq \frac{3}{4}$. Since $\mu(\mathcal{S}_n) \geq \frac{7}{8}$ for all but a density-0 set of n's, this can only occur on a set of density 0. Therefore, f is an effective dense computation of A, in contradiction to our assumptions.

Moreover, $\mathcal{A}_\Phi$ is a Π_4^A-class, as $X \in \mathcal{A}_\Phi$ iff

$$(\forall q)(\exists N)(\forall n > N)(\exists s)\Bigg[(\forall j < n)\Big[\Phi_e^X(j)[s] \downarrow = A(j) \vee \Phi_e^X(j)[s] \downarrow = 2\Big] \\ \wedge |\{j < n : \Phi_e^X(j)[s] \downarrow = A(j)\}| > n\left(1 - \frac{1}{q}\right)\Bigg].$$

Therefore, since $\mathcal{A}_\Phi$ has measure 0, $\mathcal{A}_\Phi$ contains no weakly 4-A-random. □

Unfortunately, the method of Hirschfeldt, Jockusch, Kuyper, and Schupp [9] for construction of minimal pairs appears to require a more subtle approach for effective dense computability, if it can succeed at all; much the same obstruction appears in this case as does for generic-case computability. We leave further work on this subject to a later paper.

CHAPTER 3

INTRINSIC DENSITY

3.1 Introduction

In this chapter, we turn our focus from variant computations to reconsider Jockusch and Schupp's definition of negligibility. One would expect such a definition to have interesting ties to classical computability theory. For one, a negligible set might be said to be "small", "sparse", or even "thin". Such "thinness" properties have historically been of great interest in computability; they were the focus of Post's program [22], the first attempt to construct an incomplete c.e. set, and have since proven to be of interest for unrelated reasons.

Negligibility (in the sense of asymptotic density) is closed downwards under the subset relation; any subset of a negligible set is itself negligible. It seems natural that it should be in the same family as the classical immunity properties, which provide the unifying computability-theoretic model for "thinness". However, negligibility does not lend itself to the same analysis that we apply to immunity. Choosing an alternate coding for the parameters of a membership problem is equivalent to applying a computable permutation to the underlying set, which can dramatically alter its asymptotic density. The most extreme example comes when we consider the class of infinite, co-infinite computable sets; the resulting consequences for c.e. and co-c.e. sets are essential to the remainder of this paper. We will need one standard definition of computability theory to incorporate a result of Downey, Jockusch, and Schupp: we say that a real a is left-Σ^0_2 (left-Π^0_2) if its left cut is Σ^0_2 (Π^0_2).

Proposition 3.1.1. *Suppose A is an infinite, co-infinite computable set. For any left-Σ^0_2 real a and any left-Π^0_2 real b with $0 \leq a \leq b \leq 1$, there is a computable permutation $\pi : \omega \to \omega$ such that $\pi(A)$ has lower density a and upper density b.*

Proof. We note first that there is an infinite, co-infinite computable set B with lower density a and upper density b. In fact, this is nearly a theorem of Downey, Jockusch, and Schupp

[5], which states that for any a and b meeting our preconditions, there is a computable set B with lower density a and upper density b. Unless $a = b = 0$ or $a = b = 1$, this already ensures that B is both infinite and co-infinite; if considering one of these cases, let B be the set of perfect squares or its complement, respectively.

Since the infinite, co-infinite computable sets form an orbit under computable permutations, there is a computable permutation $\pi : \omega \to \omega$ such that $\pi(A) = B$; therefore, $\pi(A)$ has lower density a and upper density b. □

Corollary 3.1.2. *If A is infinite and c.e., there is a computable permutation $\pi : \omega \to \omega$ such that $\pi(A)$ has density 1.*

Proof. Since A is infinite and c.e., A has an infinite (and co-infinite) computable subset B. By Proposition 3.1.1, there is a computable permutation $\pi : \omega \to \omega$ such that $\pi(B)$ has density 1. Since $B \subseteq A$, $\pi(B) \subseteq \pi(A)$, so $\pi(A)$ must also have density 1. □

Corollary 3.1.3. *If A is co-infinite and co-c.e., there is a computable permutation $\pi : \omega \to \omega$ such that $\pi(A)$ has density 0.*

Since any infinite c.e. set has density 1 under some computable permutation (and in fact then has a density-1 computable subset), any problem that is decidable on some infinite c.e. subset of ω is in fact effectively densely decidable if we choose the "correct" coding of the input. The corresponding coding is usually highly artificial, having little to do with the problem at hand.

In short, due to the sensitivity of asymptotic density to computable permutation, asymptotic computability is sensitive to the coding we choose for a given problem. As some of the great strengths of Turing computability come from its invariance under choice of coding, we might hope to strengthen asymptotic computability in such a way as to recover this invariance. To do so, we need to develop a stronger concept of negligibility, considering not only the upper and lower densities of a set, but those of all its images under computable permutations of ω.

In this chapter, we will study a new definition of negligibility as applied to the non-negative integers; we will spend this chapter fitting this idea into its computability-theoretic context, and then lay the foundations for further investigation into our motivating problem in Chapter 4.

In Section 3.2, we follow this approach and obtain a new pseudo-measure, *intrinsic density*, which is invariant under computable permutations of ω. We discuss various classes of sets that have intrinsic density, including the 1-random sets, which provide much of the foundation for our investigations in the rest of this chapter.

For the remainder of the chapter, we turn our focus to the new properties of intrinsic density. In Section 3.3, we begin by focusing on intrinsic density 0, the natural notion of *intrinsic negligibility*, discussing it in the context of classical computability theory. In fact, intrinsic density 0 is an immunity property, fitting naturally into the hierarchy between immunity and cohesiveness, and we determine its place in the hierarchy for both unrestricted and Δ^0_2 sets. In order to complete our description, we improve on a result of Downey, Jockusch, and Schupp [5], constructing a strongly hyperhyperimmune set with upper density at least $1 - \varepsilon$ below $\emptyset'$.

In Section 3.6, we reflect on the relation between intrinsic density and randomness, and the connection it provides between classical computability and randomness. In fact, intrinsic density provides a continuum from immunity to stochasticity, as any intrinsic density from the range $(0, 1)$ is a version of stochasticity (modulo a fixed bias), while intrinsic density 0 is a form of immunity (as discussed in the previous Section). In fact, this correspondence can be reversed to extract various strengthenings of asymptotic density from the assorted notions of stochasticity — some of which may prove fruitful topics of interest for future research.

In Section 3.4, we characterize the Turing degrees of the negligible sets. In fact, our results place non-trivial lower bounds on the degree of any set with well-defined intrinsic density, with implications for some new notions of randomness. By contrast, when we consider intrinsic *lower* density 0, a weakened notion that generalizes Post's hyperimmunity,

we find that (unlike hyperimmune sets) sets with intrinsic lower density 0 exist in every non-computable degree.

Lastly, in Section 3.5, we take advantage of the effectiveness of our proofs from Section 3.4, demonstrating that the computability-theoretic difficulty of constructing a set with intrinsic density 0 is in fact a proof-theoretic result in the program of reverse mathematics. We will show that the existence of a set with intrinsic density 0 is precisely equivalent to the disjunction of two more standard principles,

3.2 Intrinsic density

Definition 3.2.1. Let $S \subseteq \omega$. The *absolute lower density* $\underline{\boldsymbol{\rho}}(S)$ of S is

$$\underline{\boldsymbol{\rho}}(S) \coloneqq \inf_{\pi} \underline{\rho}(\pi(S)),$$

and the *absolute upper density* $\overline{\boldsymbol{\rho}}(S)$ of S is

$$\overline{\boldsymbol{\rho}}(S) \coloneqq \sup_{\pi} \overline{\rho}(\pi(S)),$$

where $\pi \colon \omega \to \omega$ is taken to vary over the set of computable permutations.

If the absolute upper and lower densities are equal, then we say that S has *intrinsic (asymptotic) density* $\boldsymbol{\rho}(S)$, where

$$\boldsymbol{\rho}(S) \coloneqq \overline{\boldsymbol{\rho}}(S) = \underline{\boldsymbol{\rho}}(S).$$

In this case, not only does S have a density, but its density is fixed under all computable permutations. We can develop analogous definitions for lower and upper densities; if $\underline{\rho}(S) = \underline{\rho}(\pi(S))$ for all computable permutations $\pi \colon \omega \to \omega$, we say that S has *intrinsic lower density* $\underline{\rho}(S)$, and similarly for *intrinsic upper density*.

If a set has intrinsic density 0, we say it is *intrinsically negligible*.

By Proposition 3.1.1, all infinite, co-infinite computable sets have absolute lower density 0 and absolute upper density 1. Thus, they are "as far as possible" from having an intrinsic density, at least in the sense that, under computable permutations, their densities range as widely as possible.

However, some might argue that 1-generic sets are further from having an intrinsic density than computable sets. It is simple to show that all 1-generic sets have lower density 0 and upper density 1. Since the class of 1-generic sets is closed under computable permutation, we can conclude that all 1-generic sets in fact have *intrinsic* lower density 0 and *intrinsic* upper density 1. This puts them "as far as possible" from having an intrinsic density, in the sense that no computable permutation can bring their upper and lower densities together.

For the rest of our work in this paper, we will focus primarily on sets that have an intrinsic density, rather than classes of sets that do not. With a few examples, we begin to establish the connections between intrinsic density and other computability-theoretic properties, and (particularly in discussing sets with intrinsic density strictly between 0 and 1) lay the groundwork for our later results.

We start with the r-cohesive and r-maximal sets. Recall that an infinite set C is *r-cohesive* if there is no computable set R such that $R \cap C$ and $\overline{R} \cap C$ are both infinite, while a c.e. set C is *r-maximal* if its complement is r-cohesive.

Theorem 3.2.2 (Jockusch, private correspondence)**.** *Every r-cohesive set has intrinsic density 0.*

Proof. We note that if a set C is r-cohesive, then its image under any computable permutation of ω is also r-cohesive; it thus suffices to prove that every r-cohesive set has density 0.

If we have a finite computable partition of ω (i.e., $\{R_0, R_1, \ldots, R_{n-1}\}$ computable and pairwise disjoint, with union ω), C must have finite intersection with all but one of these R_i, say R_j. By the finite subadditivity of upper density, $\overline{\rho}(C) \leq 0 + \overline{\rho}(C \cap R_j) \leq \overline{\rho}(R_j)$. If we take $R_i = \{kn + i \mid k \in \omega\}$, we have that $\rho(R_i) = \frac{1}{n}$, so $\overline{\rho}(C) \leq \frac{1}{n}$. Since n was an arbitrary

natural number, the upper density of C must be 0, so $\rho(C) = 0$. □

Corollary 3.2.3. *Every r-maximal set has intrinsic density 1.*

However, sets of intermediate intrinsic density (strictly between 0 and 1) provide a more versatile basis for further investigation; as such, the 1-random sets will be essential to certain constructions later in this paper.

Proposition 3.2.4. *Every 1-random set has intrinsic density* $\frac{1}{2}$.

Proof. Any 1-random set obeys the Law of Large Numbers, in the sense that it has density $\frac{1}{2}$. [21, Prop. 3.2.13] Since the class of 1-random sets is closed under computable permutations of ω, every 1-random set has intrinsic density $\frac{1}{2}$. □

We can use 1-randoms to construct sets of other intermediate intrinsic densities as well, by means of the following lemma and its corollaries.

Lemma 3.2.5. *If A has density d, and B is 1-random relative to A, then $A \cap B$ has density* $\frac{d}{2}$.

Proof. Interpreting B as a binary sequence, consider the A-computable subsequence $\widehat{B}$ selected by the rule "If $n \in A$, select $B(n)$." Since B is 1-random relative to A, we see that $\widehat{B}$ must be an unbiased sequence; in other words, $\rho(\widehat{B}) = \frac{1}{2}$.

However, by the definition of $\widehat{B}$ and asymptotic density,

$$\rho(\widehat{B}) = \lim_{n\to\infty} \frac{|(A \cap B)\upharpoonright n|}{|A\upharpoonright n|},$$

so

$$\begin{aligned}
\rho(A)\rho(\widehat{B}) &= \left(\lim_{n\to\infty} \frac{|A\upharpoonright n|}{n}\right)\left(\lim_{n\to\infty} \frac{|(A \cap B)\upharpoonright n|}{|A\upharpoonright n|}\right) \\
&= \lim_{n\to\infty} \frac{|(A \cap B)\upharpoonright n|}{n} \\
&= \rho(A \cap B).
\end{aligned}$$

Therefore, $A \cap B$ has density $\rho(A)\rho(\widehat{B}) = \frac{d}{2}$. □

Since being 1-random is invariant under computable permutation, we obtain one more pair of corollaries:

Corollary 3.2.6. *If A has intrinsic density d, and B is 1-random relative to A, then $A \cap B$ has intrinsic density $\frac{d}{2}$.*

Corollary 3.2.7. *If $\{A_1, \ldots, A_n\}$ are mutually relatively 1-random sets (i.e., each set is 1-random relative to the join of the others), then $\bigcap_{1 \leq i \leq n} A_i$ has intrinsic density 2^{-n}.*

Having established a few tools to use in controlling the intrinsic density of sets (in this paper, largely useful for the construction of counterexamples), we can now proceed to consider intrinsic density in a broader context.

3.3 Intrinsic density and immunity

As discussed in the Introduction, asymptotic density was defined as a substitute for a probability measure on a countable space. Its use in generic-case computability (and other topics) is in defining a density-0 set to be negligible, in the sense that its elements are eventually scarce. This provides one of the more practical notions of a "small" or "thin" subset of the integers, in some senses more natural than asserting that a set has no infinite c.e. subset (i.e., is *immune*).

Unfortunately, having density 0 is not computably invariant. From the perspective of computability theory, set properties that vary under computable permutation have limited applications. By addressing this one issue, having intrinsic density 0 proves to be more powerful; for example, any infinite set having intrinsic density 0 (or, in fact, any intrinsic lower density other than 1) must be immune.

Proposition 3.3.1. *Any infinite non-immune set has density 1 under some computable permutation.*

Proof. This immediately follows from Corollary 3.1.2. If S is infinite and not immune, it contains an infinite c.e. subset A. By Corollary 3.1.2, $\pi(A)$ has density 1 for some computable permutation π. Since $S \supseteq A$, $\pi(S) \supseteq \pi(A)$, so $\pi(S)$ must also have density 1. □

Corollary 3.3.2. *Any infinite set with intrinsic lower density 0, and hence, any infinite set with intrinsic density 0, is immune.*

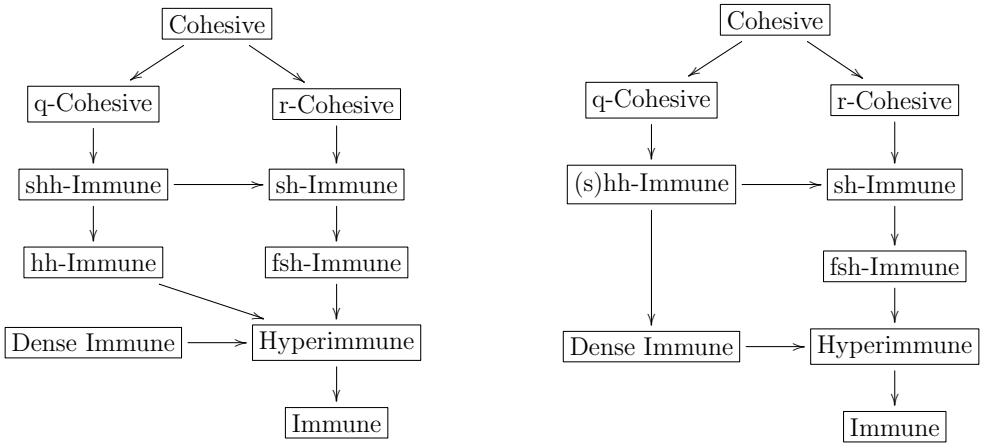
It is clear that the upper density of a set bounds the upper density of any of its subsets, so intrinsic density 0 is closed downwards under the subset relation. Since having intrinsic density 0 is a computably invariant property, closed under subsets, and implies immunity, intrinsic density 0 (here abbreviated id0) is a natural new immunity property, describing a strong notion of "thinness".

We therefore seek to determine its relation to the classical immunity properties:

Definition 3.3.3. A c.e. list of pairwise disjoint finite sets $\{D_i\}$ (indexed as finite sets, so that the sets D_i and the function $i \mapsto \max D_i$ are computable) is said to be a *strong array*. Similarly, a uniformly c.e. list of pairwise disjoint c.e. sets $\{U_i\}$ is a *weak array*. There appears to be no standard name for the intermediate concept, which we here term a *computable array*: a c.e. list of pairwise disjoint computable sets $\{C_i\}$ (indexed appropriately), with union $\bigcup C_i$ also computable.

An infinite set A is *hyperimmune* (sometimes abbreviated h-immune) if for every strong array $\{D_i\}$, there is some j such that $A \cap D_j = \emptyset$; in this case, we say that $\{D_i\}$ fails to meet A. Similarly, A is said to be *strongly hyperimmune* (sh-immune) if no computable array meets A, and *strongly hyperhyperimmune* (shh-immune) if no weak array meets A. In a slight generalization, we say A is *finitely strong hyperimmune* (fsh-immune) if no computable array $\{C_i\}$ with all C_i finite meets A, and *hyperhyperimmune* if no weak array $\{U_i\}$ with all U_i finite meets A.

It was quickly noted that a set A is hyperimmune iff no computable function dominates its principal function $p_A(n) \coloneqq (\mu x)[|S \upharpoonright x| \geq n]$; that is, for all computable functions f, we



(a) For general sets

(b) For co-c.e. sets

Figure 3.1: The graphs of the implications between the classical immunity properties; for Δ^0_2 sets, the implications are the same as in the general case, except that shh-immunity and hh-immunity become equivalent. All implications are strict, and any not shown (excepting those implied by transitivity) are false.

have $p_A(n) \geq f(n)$ infinitely often. Strengthening this, we say that an infinite set A is *dense immune* if its principal function dominates all computable functions: for all computable functions f and all sufficiently large n, we have $p_A(n) \geq f(n)$.

An infinite set A is *cohesive* if, for all c.e. sets U_i, either $A \cap U_i$ or $A \cap \overline{U_i}$ is finite. We can weaken this in a few standard ways: A is *r-cohesive* if the same property holds for computable sets C_i, or *quasicohesive* (q-cohesive) if A is a finite union of cohesive sets.

These properties are organized in a natural hierarchy of implication, shown as Figure 3.1. Chapter XI.1 of Soare [25] provides an excellent reference for this hierarchy (though focused on co-c.e. sets). We note that in the general case, the lack of implication between cohesiveness and dense immunity is witnessed by the existence of a non-high cohesive set, as constructed by Jockusch and Stephan [13]. Also, shh-immunity and hh-immunity are distinguishable in the general case, but by a remark of Cooper [2], are equivalent for Δ^0_2 sets (and thus for co-c.e. sets).

By Theorem 3.2.2, r-cohesiveness implies intrinsic density 0. This has a simple corollary, included here for completeness:

Corollary 3.3.4. *Every quasi-cohesive set has intrinsic density 0.*

Proof. As a finite union of cohesive (and thus r-cohesive) sets, any quasi-cohesive set Q is a finite union of sets of intrinsic density 0. Since density is finitely subadditive, Q must also have intrinsic density 0. □

It is slightly more interesting to note that dense immunity also implies intrinsic density 0. To show this, we note that dense immunity is computably invariant, and that a certain domination property is equivalent to having density 0.

Lemma 3.3.5. *For any infinite set $S \subseteq \omega$,*

$$\overline{\rho}(S) \coloneqq \limsup_{n\to\infty} \frac{|S\restriction n|}{n} = \limsup_{n\to\infty} \frac{n}{p_S(n)},$$

where $p_S \coloneqq (\mu x)[|S\restriction x| \geq n]$ is the principal function of S.

Proof. Consider the sequence $\{a_n\} = \left\{\frac{|S\restriction n|}{n}\right\}_{n\in\omega}$. We note that $\{b_n\} = \left\{\frac{n}{p_S(n)}\right\}_{n\in\omega}$ is an infinite subsequence — in fact, $b_n = a_{p_S(n)}$ for all $n \in \omega$ — so

$$\limsup_{n\to\infty} b_n \leq \limsup_{n\to\infty} a_n = \overline{\rho}(S).$$

The sequence a_n may increase only at n in the image of p_S (and thus at points also in the subsequence b_n), so these limits must be equal. □

Proposition 3.3.6. *An infinite set $S \subseteq \omega$ has density 0 iff its principal function dominates all linear functions. (In standard asymptotic ["Big-O"] notation, S has density 0 iff $p_S(n) \in \omega(n)$.)*

Proof. S has density 0 iff $\overline{\rho}(S) = 0$, and by the preceding lemma,

$$\overline{\rho}(S) = \limsup_{n\to\infty} \frac{n}{p_S(n)}.$$

However, for all $k > 0$, $\limsup_{n\to\infty} \frac{n}{p_S(n)} < \frac{1}{k}$ iff for all $c \in \mathbb{R}$, $p_S(n)$ dominates $kn + c$; therefore, $\overline{\rho}(S) = 0$ iff $p_S(n)$ dominates $kn + c$ for all $k > 0$ and $c \in \mathbb{R}$. □

Proposition 3.3.7. *If the set A is dense immune, $\pi(A)$ is also dense immune for any computable permutation π.*

Proof. Let π be a computable permutation, and consider a computable function f. We define

$$\widehat{f}(n) = 1 + \max_{x\in[0,f(n))} \pi^{-1}(x).$$

Since $\widehat{f}$ is a computable function, the principal function of A must dominate $\widehat{f}$; that is, $p_A(n) > \widehat{f}(n)$ for all but finitely many n. In other words, for all sufficiently large n, there are fewer than n elements of A less than $\widehat{f}(n)$.

However, by construction of $\widehat{f}$, every element of $\pi(A)$ less than $f(n)$ must come from an element of A below $\widehat{f}(n)$. Since for almost all n, there are fewer than n elements of A below $\widehat{f}(n)$, we see that $p_{\pi(A)}$ dominates f. Since both f and π were arbitrary, every computable permutation of A is dense immune. □

Corollary 3.3.8. *If $S \subseteq \omega$ is dense immune, S has intrinsic density 0.*

Proof. By Proposition 3.3.7, dense immunity is a computably invariant property. It therefore suffices to show that dense immunity implies density 0. However, this is an immediate consequence of Proposition 3.3.6, as all linear functions are bounded by computable functions, and so are dominated by the principal function of any dense immune set. □

None of the remaining standard immunity properties imply intrinsic density 0. In fact, as demonstrated by Downey, Jockusch, and Schupp [5], for every $\varepsilon > 0$, there is a strongly hyperhyperimmune set with upper density at least $1-\varepsilon$ (though none have upper density 1), constructed by Mathias forcing. We here extend their result, using a direct $\emptyset'$-computable construction, to show that we may assume these sets to be Δ^0_2. Of course, per the aforemen-

tioned remark of Cooper [2], any Δ^0_2 hh-immune set is in fact shh-immune; this apparently does not simplify the argument required, so we will make no use of this fact.

Theorem 3.3.9. *For all $\varepsilon > 0$, there is a Δ^0_2 (s)hh-immune set A with upper density at least $1 - \varepsilon$.*

Proof. We assume that ε is rational; if not, we can replace it by any rational less than ε. We will construct A as a $\emptyset'$-computable set, consulting $\emptyset'$ as an oracle during our otherwise-computable construction.

We work to satisfy the requirements:

$$P_e : (\exists n > e)[\rho_n(A) \geq 1 - \varepsilon],$$
$$N_e : (\forall i, j)\big[(i \neq j) \implies \big(U_{e,i} \cap U_{e,j} = \emptyset\big)\big] \implies (\exists k_e)\big[A \cap U_{e,k_e} = \emptyset\big],$$

where $\{U_{e,i}\}$ is a listing of the uniformly c.e. sequences of sets. (In other words, every uniformly c.e. sequence of sets is of the form $\{U_{e,i}\}_{i \in \omega}$ for some e.)

The negative requirements N_e together assert that every weak array fails to meet A; this is the definition of shh-immunity.

Our negative requirements should, in principle, be simple to satisfy. We simply omit a set from each weak array of small lower density, thus leaving us with a set A with high upper density. There are slight complications in ensuring that taking all of these requirements still cannot force A's upper density to fall below $1 - \varepsilon$, but these are easily addressed. After all, at any given point n, only q disjoint sets can have partial density exceeding $\frac{1}{q}$; therefore, for any weak array $\{U_{e,i}\}_{i=0}^{\infty}$, there must be some $U_{e,k}$ with lower density less than $2^{-e}\varepsilon$.

The trouble comes in attempting to identify the $U_{e,k}$ in question. $\emptyset'$ is incapable of resolving whether a c.e. set has lower density below some bound; in fact, this problem is Σ^0_2-hard, as it would allow us to distinguish finite c.e. sets from total c.e. sets. Therefore, we cannot search directly for such a U_k in our weak array, and must use more indirect methods.

Towards this end, we will make heavy use of the following sentence, for varying values of

e, n, and r:

$$(\exists x_n > x_{n-1} > \cdots > x_1 \geq r)(\exists s)(\exists t > s) \\ \big[t \in C_{e-1,s} \wedge (\forall i \leq n)\big[\rho_t(U_{e,x_i}) \geq d_e\big]\big]. \tag{*}$$

$C_{e-1,s}$ will be defined in the course of our construction, but is c.e.. The $U_{e,i}$'s are taken from our listing of uniformly c.e. sequences of sets. As all sets involved are c.e., and we only ask whether a c.e. set has partial density exceeding some lower bound, (*) is a Σ^0_1 sentence, and thus decidable by $\emptyset'$.

Putting (*) in context, we understand it to say that there are n elements of our weak array $\{U_{e,i}\}$, not including any with index less than r, which all have high partial density (exceeding d_e) at a single point $t > s$, where t is chosen from some c.e. set C_{e-1} of possibilities. Clearly, this sentence is vacuously true for $n = 0$, and (presuming our $U_{e,i}$'s to be disjoint) necessarily false for $n > \frac{1}{d_e}$. Therefore, for any fixed r and e, the maximum n for which this sentence holds is computable in $\emptyset'$ by a simple bounded search; let us refer to it as N. If we have N such elements of a weak array, we refer to them together as a maximal tuple for that array under the conditions C_{e-1}, r, and e.

If we have a maximal tuple for our weak array, and $U_{e,k}$ is not among its members, then we know that $\rho_t(U_{e,k}) < d_e$ for some $t \in C_{e-1}$ with $t > s$. Otherwise, (*) would be satisfied with $n = N + 1$. This will be our primary tool for controlling the lower density of the sets we omit as we build A to avoid meeting the weak array $\{U_{e,i}\}_{i=0}^{\infty}$.

Organization:

As we combine multiple negative requirements, we allow finite injury of each negative requirement by higher-priority requirements, though never revoking any previous decisions as to whether $s \in A$. We activate the requirements in order of decreasing priority, activating at most one at each stage. At stage s, each active requirement independently decides whether to allow s into A; we put s into A if none of these requirements object. For convenience, we denote $A\upharpoonright s$ by A_s.

Each requirement N_e will maintain a c.e. set $S_{e,s}$ of elements such that, if $A \cap S_{e,s} = \emptyset$, then N_e will be satisfied. For internal reference, we will also keep track of $k_{e,s}$, which determines which element of the weak array $\{U_{e,i}\}$ we are actually restricting out of A. Lastly, we will maintain a c.e. set $C_{e,s}$ of locations where the partial density of $U_{e,k_{e,s}}$ is known to be strictly less than $2^{-e-3}\varepsilon$, while guaranteeing that $C_{e,s} \subseteq C_{e-1,s}$ at all stages s.

N_e's basic goal is to prevent the weak array $\{U_{e,i}\}$ from meeting A, while ensuring that the lower density of its restricted set does not exceed $d_e = 2^{-e-3}\varepsilon$. To do so, N_e will repeatedly consult $\emptyset'$ regarding (*). In context, we can now see that we choose $t \in C_{e-1,s}$ to ensure that the density of the set we omit for N_e falls below d_e at the same time as the densities of the previously-chosen sets fall below their critical values; this will make certain that the density of our set A rises above its goal of $1 - \varepsilon$.

Module for N_e:

On activation at stage s: We first consult $\emptyset'$, asking whether the sets $\{U_{e,i}\}$ are in fact pairwise disjoint (i.e., $\{U_{e,i}\}$ is a weak array). If not, then N_e is trivially satisfied. In this case, N_e will never restrict anything out of A; it simply maintains $S_{e,t} = \emptyset$ and $C_{e,t} = C_{e-1,t}$ at all stages $t \geq s$, while voting to allow all elements into A.

If $\{U_{e,i}\}$ is a weak array, we define $k_{e,s-1}$ to be the least k such that $U_{e,k} \upharpoonright s = \emptyset$. We then set $S_{e,s-1} = U_{e,k_{e,s-1}}$, and let $C_{e,s-1} = \emptyset$, as we do not yet know of any locations where $\rho_t(S_{e,s}) < d_e$.

At stage s: We assume that $A_s = A \upharpoonright s$ has already been determined, and consider only whether to allow s into A. Before making this decision, we must first determine whether we can still believe that we can restrict $S_{e,s-1}$ out of A while keeping $\overline{\rho}(A)$ close to 1. In fact, we want to verify that $S_{e,s-1}$ will again appear to have partial density less than d_e at some point $t > s$ where the partial density of $S_{i,s}$ (for all $i < e$) is also small.

If $C_{e,s-1} \cap (s, \infty) \neq \emptyset$ (a $\emptyset'$-computable question), we have already verified this at some previous stage. We simply define $S_{e,s} = S_{e,s-1}$, set $k_{e,s} = k_{e,s-1}$, and let $C_{e,s} = C_{e,s-1}$. We then allow s into A iff $s \notin S_{e,s}$.

If $C_{e,s-1} \subseteq [0,s]$, though, we must attempt to verify that the partial density of $S_{e,s-1}$ will fall below d_e at some point in the future. We know that $\emptyset'$ cannot answer this question directly, as it cannot determine whether a c.e. set will ever have partial density less than some critical value. We instead use (*) to attack from a different angle. We will need to reference $k_{e,s-1}$ several times in the remainder of the procedure; for simplicity's sake, we will abbreviate it by $k = k_{e,s-1}$.

We first determine $n_{e,s}$, the greatest value of n for which (*) holds with $r = k$. Since at most $\lfloor 1/d_e \rfloor$ disjoint sets can have partial density exceeding d_e at the same location t, this is a bounded search on a parameter of a Σ^0_1 statement; thus, $\emptyset'$ suffices to compute $n_{e,s}$.

We then ask $\emptyset'$ whether (*) holds with $n = n_{e,s}$ and $r = k+1$. If so, then we have a maximal tuple (within the array $\{U_{e,i}\}$ for $i > k$) in which every set has high partial density at the same point $t > s$. Since we cannot add $U_{e,k}$ to this collection, we must have $\rho_t(U_{e,k}) < d_e$. We define $C_{e,s}$ to be the set of all $t > s$ for which there is such a collection (along with all $t \leq s$ for which $\rho_t(S_{e,s}) < d_e$), set $k_{e,s} = k_{e,s-1}$ and $S_{e,s} = S_{e,s-1}$, and allow s into A iff $s \notin S_{e,s}$. If this case occurs immediately following injury or initialization of N_e at stage $s-1$, we say that s was a "recovery stage" for N_e; otherwise, we deactivate all lower-priority requirements N_i $(i > e)$, as $C_{e,s}$ has changed.

Otherwise, the (*) does not hold with $n = n_{e,s}$ and x_1 strictly greater than k. In this case, we have no way to verify that the density of $U_{e,k}$ again drops below d_e, and so consider N_e to be injured. We vote to allow s into A, and deactivate all lower-priority requirements N_i $(i > e)$. We then effectively reset our procedure for N_e; we define $k_{e,s}$ to be the least $k > k_{e,s-1}$ such that $U_{e,k} \cap [0,s] = \emptyset$, set

$$S_{e,s} = \big(S_{e,s-1} \cap [0,s)\big) \cup U_{e,k},$$

and let $C_{e,s} = \emptyset$.

Verification of the basic module:

Suppose that the module for N_e is at some point activated and never again deactivated (i.e., $C_{e-1,s}$ does not change at any later stage s). We assume that $\{U_{e,i}\}$ is in fact a weak array; if it is not, N_e is trivially satisfied, $S_{e,s} = \emptyset$ has partial density identically 0 for all s, and $C_{e,s} = C_{e-1,s}$ does not change at any later stage s.

By the construction of $S_{e,s}$, we know that $S_e = \lim_{s\to\infty} S_{e,s}$ exists, and consists of all elements restricted out of A by N_e. We will show that there is some stage s_0 at which C_{e,s_0} is infinite, thus ensuring that $C_{e,s}$ will not change at any later stage and preventing future injury to N_e. This will also guarantee that $S_e = S_{e,s_0}$ and $k_e = \lim_{s\to\infty} k_{e,s} = k_{e,s_0}$.

Given such an s_0, since A does not intersect $S_{e,s_0} \supseteq U_{e,k_e}$, we have satisfied N_e. Furthermore, the partial density of S_e approaches that of U_{e,k_e}, as the sets agree on all $x \geq s_0$; therefore, if the partial density of U_{e,k_e} drops below d_e infinitely often, the partial density of $S_e = S_{e,s_0}$ must be less than $2d_e$ at all but finitely many of the same points.

We note that the sequence $\{n_{e,s}\}$ is nonincreasing, as we monotonically reduce the set of witnesses for (*) at successive stages s. In fact, the sequence must decrease each time $C_{e,s}$ changes (except at recovery stages); this can only happen when we have run out of witnessing collections of size $n_{e,s-1}$. As for recovery stages, they can only occur immediately after initialization of N_e, or immediately after an injury to N_e; since injuries cause $n_{e,s}$ to decrease, any recovery stage is still associated with a corresponding decrease in $n_{e,s}$. Since for $n = 0$, (*) is vacuously true, $n_{e,s}$ is always a non-negative integer and so cannot decrease infinitely often. Therefore, there must be some stage s_0 such that $C_{e,t} = C_{e,s_0} \neq \emptyset$ for all $t > s_0$, which is only possible if C_{e,s_0} is infinite.

Lastly, the module must force $A \cap S_e = \emptyset$. Whenever we choose a new $k_{e,s}$, we always choose a value k such that $A_s \cap U_{e,k} = \emptyset$, and redefine $S_{e,s}$ accordingly to remain disjoint from A_s. When we keep the same k, we allow elements into A iff they are not in $S_{e,s}$. Therefore, as long as N_e is active, we are assured that $A_s \cap S_{e,s} = \emptyset$ for all s; since $S_e \upharpoonright s = S_{e,s} \upharpoonright s$, we will always have $A \cap S_e = \emptyset$.

Construction of A:

At stage 0, begin by activating N_0.

At stage s, check whether $\rho_s(A) \geq 1 - \varepsilon$. If so, determine the highest-priority inactive requirement N_e. If N_e was deactivated in stage $s - 1$, do not activate any requirements; otherwise, activate N_e. (This delay in N_e's reactivation ensures that N_e is not activated during a recovery stage for N_{e-1}.)

Next, consult all active requirements in priority order. If any restrict s out of A, we declare that $s \notin A$; if all allow s to enter A, we put s into A.

Verification:

Nothing can deactivate N_0, so N_0 is permanently activated. By the correctness of the basic module, if the module for N_e is permanently activated, N_e will be satisfied. Furthermore, there is some stage s_0 after which $C_{e,s}$ does not change, so that N_{e+1} will never again be deactivated. Therefore, as long as there are infinitely many stages at which we activate some inactive requirement, every module will be permanently activated at some point, and thus every N_e will be satisfied.

Suppose, towards a contradiction, that some requirement is never permanently activated. Let N_e be the highest-priority such requirement, so that only modules N_0 through N_{e-1} are permanently activated. We consider the construction at stage s_0, after the last such module has been permanently activated and $C_{e-1,s}$ has stopped changing (and is infinite).

At this stage, N_e can never again be deactivated, so since N_e is not permanently activated, N_e must never again be activated. This can only be because the construction will never reach another stage where it activates the highest-priority inactive requirement; therefore, it must be that $A \cap (s, \infty)$ contains all elements except those in $S = \bigcup_{i<e} S_i$, and $\rho_t(A) < 1 - \varepsilon$ for all $t > s$. For all sufficiently large n, we have that $\rho_n(A) < 1 - \varepsilon$ implies $\rho_n(S) > \frac{1}{2}\varepsilon$; thus, $\rho_n(S) > \frac{1}{2}\varepsilon$ for all sufficiently large n.

However, $\rho_n(S) = \rho_n(\bigcup_{i<e} S_i) \leq \sum_{i<e} \rho_n(S_i)$. Recall that for all but finitely many

$n \in C_{e-1}$, we have $\rho_n(S_i) < 2d_e = 2^{-n-2}\varepsilon$, so

$$\rho_n(S) \leq \sum_{i<e} 2^{-i-2}\varepsilon < \frac{1}{2}\varepsilon.$$

Since C_{e-1} is infinite, this is a contradiction; therefore, all modules N_e must be permanently activated eventually.

Finally, since every module is eventually activated, we must activate a new module infinitely often. This can only happen if $\rho_s(A) \geq 1 - \varepsilon$ infinitely often, so every requirement P_e is also satisfied; A must have upper density at least $1 - \varepsilon$. □

We have yet to consider implications in the other direction; what immunity properties are implied by intrinsic density 0? The first such result is simple; as established above in Corollary 3.3.2, intrinsic density 0 at least implies immunity for infinite sets.

On the other hand, we already know that hyperimmunity (even shh-immunity) does not imply intrinsic density 0. We can further prove that a set of intrinsic density 0 need not be hyperimmune; we can construct Δ^0_2 counterexamples, and in fact will build a counterexample below every 1-random set.

Theorem 3.3.10. *For every 1-random set R, there is an infinite set $A \leq_{\mathrm{T}} R$ with intrinsic density 0 that is not hyperimmune.*

Proof. Suppose that $K(R\upharpoonright n) \geq n - c$ for all n.

By van Lambalgen's Theorem [27], given a 1-random set R, there exists a uniformly R-computable sequence of sets $\{R_j\}_{j\in\omega}$ that are mutually relatively 1-random. In fact, defining $\widehat{R}_j = \bigoplus_{i<j} R_i$, we have that

$$K^{\widehat{R}_j}(R_j \upharpoonright n) \geq n - d_j$$

for all n, where d_j is uniformly computable from j and c; this can be shown by a simple inspection of a proof of van Lambalgen's Theorem.

Let $A_0 = R_0$. Since A_0 is 1-random, it has intrinsic density $\frac{1}{2}$. Given d_0 and using the incompressibility of R_0, we can compute k_0 such that $|R_0 \upharpoonright k_0| \geq 1$, ensuring that $A_0 \cap [0, k_0) \neq \emptyset$. Since k_0 is computable, we can use $[0, k_0)$ as the first partition in a weak array that will witness that the set we construct A is not hyperimmune.

We then define

$$A_1 = A_0 \cap ([0, k_0) \cup R_1).$$

Since $A_1 =^* R_0 \cap R_1$, and R_0 and R_1 are mutually relatively 1-random, A_1 must have intrinsic density $\frac{1}{4}$ by Corollary 3.2.7. Using d_0 and d_1 along with the incompressibility of R_1 (relative to R_0), we can compute k_1 such that $|A_1 \cap [k_0, k_1)| \neq \emptyset$.

Repeating this process, we see that $A = \bigcap_j A_j$ is computable in R, since $A \upharpoonright k_j = A_j \upharpoonright k_j$. For all j, we have that $A \cap \left[k_{j-1}, k_j\right) \neq \emptyset$, so A is infinite. Furthermore, since the k_j's are uniformly computable from c, an integer, this partition of ω is in fact computable, demonstrating that A is not hyperimmune.

Finally, $A \subseteq A_j$ for all j. Since $A_j =^* \bigcap_{i \leq j} R_i$, and the R_i's are mutually relatively 1-random, A_j has intrinsic density 2^{-j-1}; therefore, A must have intrinsic density 0. □

As a convenient side effect, this theorem immediately gives us some information on the Turing degrees of infinite sets with intrinsic density 0: such sets exist below every 1-random Turing degree, but cannot be computable. Among other things, this implies that there are infinite id0 sets in non-computable Δ^0_2, low, and even hyperimmune-free degrees (where our construction of a non-hyperimmune A becomes rather superfluous, though it at least ensures that A is infinite).

It still eliminates any hopes we might have of further positive implications between the immunity properties and intrinsic density, as we will discuss in our summary below. By falling back to intrinsic lower density, we can recover one more positive implication, as shown by Jockusch in private correspondence.

Theorem 3.3.11. *Every hyperimmune set has intrinsic lower density 0.*

Proof. Since hyperimmunity is computably invariant, it suffices to show that every hyperimmune set has lower density 0.

Suppose A is hyperimmune. Consider the strong array

$$D_n = [n!, (n+1)!)\,.$$

Since A is hyperimmune, $A \cap D_n = \emptyset$ for infinitely many n. For all such n, we have that $|A\upharpoonright(n+1)!| \leq n!$; therefore, $\rho_{(n+1)!}(A) \leq \frac{1}{n}$. Since this occurs infinitely often, we conclude that $\underline{\rho}(A) = 0$. □

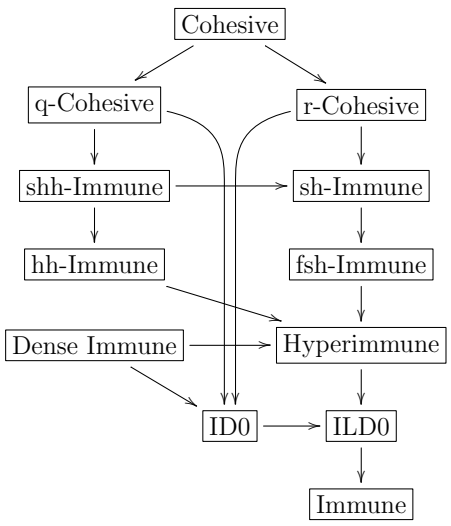
The above results, along with earlier work [5, 12], will suffice to disprove all other potential implications between intrinsic density 0, intrinsic lower density 0, and the standard immunity properties.

We first repeat, per Jockusch and Schupp [12], that any 1-generic set has lower density 0 and upper density 1; since 1-genericity is computably invariant, 1-generics in fact have intrinsic lower density 0 and intrinsic upper density 1. Therefore, intrinsic lower density 0 does not imply intrinsic density 0, even for Δ^0_2 sets.

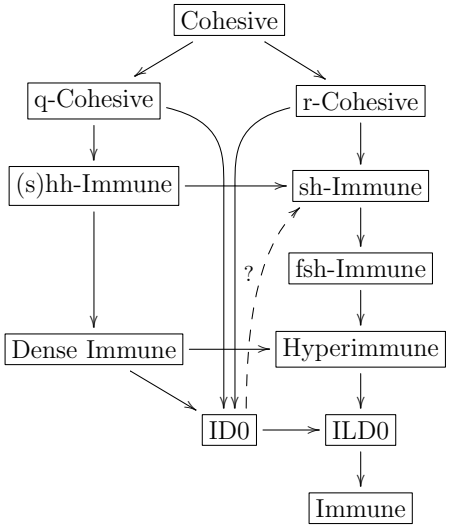
In addition, all 1-random sets are immune; otherwise, there would be a 1-random R with an infinite computable subset, which admits a trivial computable martingale that succeeds on R. Since 1-randoms have intrinsic density $\frac{1}{2}$, immunity does not imply intrinsic lower density 0, even for Δ^0_2 sets.

Lastly, Theorem 3.3.9 above demonstrates that for every $\varepsilon > 0$, there is a shh-immune set (in fact, a Δ^0_2 hh-immune set) with upper density at least $1 - \varepsilon$. In particular, shh-immunity does not imply intrinsic density 0, even for Δ^0_2 sets.

Combining these counterexamples with our results above, we exhaust all possible implications between intrinsic density 0, intrinsic lower density 0, and the standard immunity properties. The graph of the resulting implications for infinite sets is shown in Figure 3.2; all implications depicted are strict, and counterexamples are discussed above for all arrows



(a) For general sets



(b) For co-c.e. sets

Figure 3.2: The graph of implications between the classical immunity properties and intrinsic density 0. The single dashed arrow indicates an open implication. Again, for Δ^0_2 sets, shh-immunity and hh-immunity become equivalent; all other implications are as depicted for general sets. (We abbreviate intrinsic [lower] density 0 for infinite sets by I[L]D0.)

not present in the diagram.

Unfortunately, in the c.e. case (well-studied due to Post's Program), the majority of our proofs of failures of implication collapse. Since hh-simplicity does imply dense simplicity for c.e. sets, it seems unlikely that our proof method from Theorem 3.3.9 will help separate the higher immunity properties from intrinsic density 0. In fact, most of our other failures of implication are exhibited by 1-generics or derived from 1-randoms, examples that are inherently not c.e. We will recover one of these in Section 3.4; by Corollary 3.4.8, there are hypersimple sets with lower density 0, so hypersimplicity does not imply intrinsic density 1.

This leaves the c.e. diagram incomplete, with one family of questions remaining:

Open Question 3.3.1. Is there an infinite c.e. set with intrinsic density 1 that is not hypersimple? Not (f)sh-simple?

3.4 Intrinsic density and computability

Since we've established (by Proposition 3.1.1) that infinite co-infinite computable sets cannot have intrinsic density, we might begin to ask precisely which Turing degrees compute effectively negligible sets. After all, all such sets are intrinsically asymptotically computable in some sense, so this would seem likely to help us develop the appropriate notions of intrinsic asymptotic computation. Moreover, most other notions of immunity carry some implication of computability-theoretic strength, and their study has proven extremely rich; it would be surprising if effective negligibility was the exception to the rule.

Of course, we have already established some upper bounds on the information content required for a set to have intrinsic density 0. As every r-cohesive set has intrinsic density 0 (3.2.2), these sets exist in every cohesive degree, and hence in every high degree. Moreover, in Theorem 3.3.10, we found sets with intrinsic density 0 below every 1-random set, and thus computable in assorted low or even hyperimmune-free degrees. Though not one of the more typical combinations found in computability, there is in fact a connection between these constructions, centering on a notion introduced by Kjos-Hanssen, Merkle, and Stephan [15] in their study of eventually different functions (though we use the name introduced in Downey and Hirschfeldt [3]).

Definition 3.4.1. A set A is *weakly computably traceable* if there is a computable function h such that for all $f \leq_{\mathrm{T}} A$, there is a computable sequence of finite sets V_n with $|V_n| \leq h(n)$ for all n and $f(n) \in V_n$ for infinitely many n; that is, if we can infinitely often guess the value of $f(n)$ using a computable guessing strategy limited to at most $h(n)$ guesses.

By a result of Kjos-Hanssen, Merkle, and Stephan [15] (their Theorem 5.1), a set A is weakly computably traceable (WCT) iff it has neither high nor DNC degree; that is, iff it computes neither a dominant function [18] nor a diagonally non-computable function. Thus, weak computable traceability is a property of Turing degrees expressing computability-theoretic weakness.

Since all 1-random sets compute a DNC function, and all r-cohesive sets have either high or DNC degree [13], all of our prior constructions of a set with intrinsic density 0 were built below non-WCT sets. We can now show that this was no coincidence.

Theorem 3.4.2. *Every infinite set A that is weakly computably traceable has upper density 1 under some computable sampling, and thus has absolute upper density 1.*

Proof. By the same equivalence of Kjos-Hanssen, Merkle, and Stephan [15], A is weakly computably traceable iff for all $f \leq_T A$, there is a total computable function h such that $h(n) = f(n)$ for infinitely many n.

Let $f(n)$ code $p_A(j)$ for all $j < n!$; specifically, take $f(n) = A\upharpoonright p_A(n!)$. Clearly $f \leq_T A$, so there is a total computable h with $h(n) = f(n)$ for infinitely many n.

We define a total computable injection g by assigning values $g(j)$ in increasing order of j. If $j \in [(n-1)!, n!)$, define $g(j)$ to be the position of the j-th 1 in the string $h(n)$, unless this value is already assigned to some $g(i)$ with $i < j$; in that case, we instead define $g(j)$ to be the least value not assigned to any earlier $g(i)$.

For any n where $h(n) = f(n)$, we then have $g(j) \in A$ for all $j \in [(n-1)!, n!)$, unless the requisite value was already assigned at that stage. In any event, $g([0, n!))$ contains at least $n! - (n-1)!$ elements of A, so $\rho_{n!}(g^{-1}(A)) \geq 1 - \frac{1}{n}$. Since this occurs for infinitely many n, we conclude that g samples A with upper density 1, and thus (by Lemma 3.6.2) that A has absolute upper density 1. □

Corollary 3.4.3. *Every infinite co-infinite set A that is weakly computably traceable has absolute upper density 1 and absolute lower density 0.*

Proof. Apply Theorem 3.4.2 to both A and $\overline{A}$. □

What's more, we can generalize our prior constructions to build a set with intrinsic density 0 in every other degree.

Theorem 3.4.4. *Every set A that is not weakly computably traceable computes a set with intrinsic density 0.*

Proof. By definition, A is not weakly computably traceable iff for all computable orders h, there is some $f \leq_T A$ such that for no computable function $g(n)$ do we have $|D_{g(n}| \leq h(n)$ for all n and $f(n) \in D_{g(n)}$ infinitely often.

Take $h(n) = n^2$ (or, indeed, any computable superlinear function), and let $f \leq_T A$ be as above. We claim that $G_f = \{\langle n, f(n)\rangle : n \in \mathbb{N}\}$, the graph of f, has intrinsic density 0.

Suppose not; in particular, suppose that G_f has upper density greater than $\frac{1}{q}$ under some computable permutation π. Thus, G_f has partial density exceeding $\frac{1}{q}$ in the first s positions for infinitely many s. For such s, we have that $\pi([0, s))$ contains at least $\frac{s}{q}$ elements of G_f, and thus must contain $\langle m, f(m)\rangle$ for some $m \geq \frac{s}{q} - 1$. Therefore, for infinitely many m, we have that $\pi([0, (m + 1)q))$ contains $\langle m, f(m)\rangle$.

For all n, define $D_{g'(n)} = \{y : \langle x, y\rangle \in \pi([0, (n + 1)q))\}$. For all sufficiently large n, $h(n) \geq (n+1)q$; thus, there is a computable function g such that $D_{g(n)} \leq h(n)$ for all n and $g(n) = g'(n)$ for all sufficiently large n.

However, as noted above, $\pi([0, (m + 1)q))$ contains $\langle m, f(m)\rangle$ for infinitely many m. As this implies that $f(m) \in D_{g(m)}$ for infinitely many m, this contradicts our choice of f. Therefore, we conclude that G_f has intrinsic density 0. □

By a result of Jockusch [11], the class of Turing degrees containing sets with intrinsic density 0 is upwards closed, as this class is closed under subset and contains an arithmetic member (for instance, any Δ^0_2 cohesive set). Combining this observation with Corollary 3.4.3 and Theorem 3.4.4, we obtain the following corollary:

Corollary 3.4.5. *The Turing degrees containing an infinite set of intrinsic density 0 are precisely those that are not weakly computably traceable; that is, those that are either high or DNC.*

Since all weakly computably traceable sets have absolute upper density 1 by Theorem 3.4.2, this gives a 0-1 law for absolute upper density:

Corollary 3.4.6. *A Turing degree contains no set with absolute upper density 0 iff all its sets have absolute upper density 1.*

In fact, we can broaden our results slightly to all sets with defined intrinsic density, since Corollary 3.4.3 states that no non-trivial weakly computably traceable set has any defined intrinsic density.

Corollary 3.4.7. *The Turing degrees containing infinite co-infinite sets with intrinsic density are precisely those that are not weakly computably traceable; that is, those that are either high or DNC.*

By Arslanov's completeness criterion, any DNC c.e. set is in fact Turing-equivalent to $\emptyset'$; therefore, each non-high co-infinite c.e. set has absolute lower density 0, as its complement is weakly computably traceable.

On the other hand, as there is a dense simple (in fact, maximal) set in every high c.e. degree (Martin [18]), every high c.e. degree contains a c.e. set with intrinsic density 1, by Corollary 3.3.8. Therefore, the Turing degrees computing a c.e. set of intrinsic density 1 are precisely the high c.e. degrees.

Since there are non-high hypersimple sets (in fact, every non-computable c.e. degree contains a hypersimple set), this answers one of our last questions from Section 3.3:

Corollary 3.4.8. *There is a hypersimple set with lower density 0.*

Proof. Let A be any non-high hypersimple set. Since A is c.e. and non-high, it is weakly computably traceable, and therefore has lower density 0 under some computable sampling. Permuting A by the computable permutation constructed in the proof of Lemma 3.6.2, and noting that hypersimplicity is computably invariant, we obtain a hypersimple set with lower density 0. □

By contrast, intrinsic lower density is much simpler to control. In fact, we find that the traditional construction of an immune set in every non-computable degree in fact yields something slightly stronger:

Theorem 3.4.9. *Let S be the set of prefixes of a set A. If A is not computable, then S has intrinsic lower density 0.*

Proof. Suppose $S = \{A\upharpoonright n : n \in \mathbb{N}\}$, the set of prefixes of A, does not have intrinsic lower density 0. By definition, there exists some total computable injection φ_e and some integers q and N such that $\rho_n(\varphi_e^{-1}(S)) > \frac{1}{q}$ for all $n > N$.

Given e, q, and N, we construct the computable binary tree T as follows:

T begins as a full tree up to height N. For strings σ of length $n > N$, we put σ into T if and only if its prefixes are in T and $\varphi_e([0, 2qn))$ contains at least n strings extending σ.

As no two distinct strings of the same length can share an extension, and since $\sigma \in T$ implies that $\varphi_e([0, 2q|\sigma|))$ contains at least $|\sigma|$ extensions of σ, we see that T has width at most $2q$ at all heights $n > N$.

By assumption, φ_e samples the prefixes of A with partial density always exceeding $\frac{1}{q}$ beyond a point N; therefore, for $n > N$, $\varphi([0, 2qn))$ must contain at least $2n$ prefixes of A, and so must include at least n extensions of $A\upharpoonright n$. Thus, A must be a path on T.

Since T is a computable tree with bounded width, A is computable. □

Again, by the aforementioned result of Jockusch [11], the class of Turing degrees containing sets with intrinsic lower density 0 is upwards closed. Therefore, our theorem becomes a characterization of this class:

Corollary 3.4.10. *The Turing degrees containing an infinite set with intrinsic lower density 0 are precisely the non-computable degrees.*

Corollary 3.4.11. *The Turing degrees containing a co-infinite set with intrinsic upper density 1 are precisely the non-computable degrees.*

3.5 Intrinsic density in reverse mathematics

In this section, we make common reference to an arbitrary model of second-order arithmetic,

$$\mathcal{M} = (M, S, +, \cdot, 0, 1),$$

where M and S are, respectively, the first- and second-order parts of the structure.

We note that the proofs of Theorems 3.4.2 and 3.4.4, as given above, appear sufficiently constructive to hold in $\mathbf{RCA}_0$, demonstrating in a vague sense that the existence of a set that is not weakly computably traceable should be reverse-mathematically equivalent to the existence of a set with intrinsic density 0 over $\mathbf{RCA}_0$.

To make this more precise, we must use the notion of a weakly-represented family of functions, as introduced by Zhang and Stephan [29].

We recall their definitions:

Definition 3.5.1 (Weakly-represented partial functions)**.** A partial function f is weakly represented by the set A if all of the following conditions hold:

- [Representation] For all x and y, $f(x) \downarrow = y$ iff there is some z such that $\langle x, y, z\rangle \in A$. We say A *witnesses* that $f(x)$ converges to y.
- [Consistency] If $\langle x, y, z\rangle$ and $\langle x, y', z'\rangle$ are both in A, then $y = y'$.
- [Monotonicity] If $\langle x, y, z\rangle \in A$, then for all $z' > z$, we also have $\langle x, y, z\rangle \in A$.
- [Downward closure] If A witnesses that $f(x)$ converges, then it also witnesses that $f(t)$ converges for all $t < x$.

By convention, for f weakly represented by A, we say that $f(x)$ *converges to y by step z* ($f(x)[z] \downarrow = y$) if $y < z$ and $\langle x, y, z\rangle \in A$. Along the same line, we say that $f(x)[s]$ converges iff it converges to some $y < s$; this restriction ensures that the question of whether $f(x)[s]$ converges is decidable in our representation of f.

Definition 3.5.2 (Weakly-represented families)**.** A class of partial functions $\{f_e\}_{e\in M}$ is weakly represented in our model iff S contains a uniform family of sets $\{A_e\}_{e\in M}$ (represented by $A = \{\langle e, x\rangle : x \in A_e\} \in S$) such that A_e weakly represents f_e.

A class of total functions $\mathcal{F}$ is weakly represented in our model iff S contains $\mathcal{F}$ and a weakly-represented class of partial functions $\{f_e\}_{e\in M}$ such that a total function f is in $\mathcal{F}$ iff $f = f_e$ for some $e \in M$.

Restricting ourselves to 0-1 functions in the latter case naturally provides the idea of a weakly-represented family of sets.

These definitions enable us to discuss the subset of total functions within a larger class of p.c. functions. For instance, the family of all computable functions (or sets) is weakly representable in $\mathbf{RCA}_0$.

Zhang and Stephan defined these notions to formulate, as reverse-mathematical principles, the many concepts from classical computability theory which naturally address the class of total functions, such as dominating functions (their **DOM**) or cohesive sets (**COHW**).

Statement 3.5.3 (**DOM**)**.** For every weakly-represented family of total functions $\mathcal{F}$, there is a function g such that, for each $f \in \mathcal{F}$, there is some $b \in M$ such that $g(x) > f(x)$ for all $x > b$.

Statement 3.5.4 (**COHW**)**.** For every weakly-represented family of sets $\mathcal{F}$, there exists an $\mathcal{F}$-cohesive set.

It should be noted that we can also give an alternate form of DNR, equivalent to the standard form over $\mathbf{RCA}_0$:

Statement 3.5.5 (**DNRW**)**.** For every weakly-represented family of (partial) functions $\mathcal{F} = \{f_e\}_{e\in M}$, there exists a function F such that $F(e) \neq f_e(e)$ for all $e \in M$ such that $f_e(e)\downarrow$.

Theorem 3.5.6. *Over* $\mathbf{RCA}_0$, **DNR** *and* **DNRW** *are equivalent.*

Proof. To see that **DNRW** implies **DNR**, we note that though we cannot weakly represent the standard listings of partial functions $\left\{\varphi_e^A\right\}$ (since φ_e^A's domain need not be an initial segment of M), we can weakly represent the family of partial functions $\left\{f_e^A\right\}$ given by $f_e^A(n) = \varphi_e(e)$. Any function diagonally disagreeing with $\left\{f_e^A\right\}$ must in fact be DNR for A.

The converse implication is also relatively straightforward. Fixing some weakly-represented family of partial functions $\{f_e\}$ uniformly computable from A, we pass to another family, still uniformly computable from A:

$$g_{2e}(2n) = g_{2e}(2n+1) = f_e(n),$$
$$g_{2e+1}(n) = \varphi_e^A(n).$$

Since the f_e's were uniformly computable from A, this family $\{g_k\}$ is an *effective* universal listing of partial A-computable functions; therefore, by **DNR**, there is some G such that if $g_k(k)\downarrow$, then $G(k) \neq g_k(k)$. In particular, if $f_e(e) = g_{2e}(2e)\downarrow$, then $G(2e) \neq f_e(e)$. Thus, defining $F(e) = G(2e)$, we see that our original family $\{f_e\}$ satisfies **DNRW** via F. □

Similarly, we can state the existence of a set of intrinsic density 0 as a reverse-mathematical principle, as follows:

Statement 3.5.7 (ID0). For every weakly-represented class of total functions $\mathcal{F}$, there exists a set A such that every injective $f \in \mathcal{F}$ samples A with density 0. That is, taking $\widehat{\mathcal{F}}$ to be the weakly-represented class of total injections contained in $\mathcal{F}$,

$$(\exists A)\Big(\forall f \in \widehat{\mathcal{F}}\Big)\Big[\rho(f^{-1}(A)) = 0\Big].$$

Combining our proofs of Theorems 3.4.2 and 3.4.4 with the proof of Theorem 5.1 from Kjos-Hanssen, Merkle, and Stephan [15], we can factor out all intermediate results and obtain proofs obviously holding over $\mathbf{RCA}_0$. In fact, we will show that **ID0** is equivalent to the disjunction of **DNRW** and **DOM**:

Statement 3.5.8 (DNRW∨DOM). For every weakly-represented family of (partial) functions $\mathcal{F} = \{f_e\}_{e \in M}$, there exists either a function F dominating all total functions in $\mathcal{F}$ or a function g such that $g(e) \neq f_e(e)$ for all $e \in M$.

All proofs would hold if we restrict ourselves to classes containing only total functions, avoiding the complication of weak representation — but in this case, the result is trivial, as all three simplified principles are simply true in RCA_0.

Theorem 3.5.9. $\mathbf{RCA}_0 + \mathbf{ID0} \models \mathbf{DNRW} \vee \mathbf{DOM}$.

Proof. Let $\mathcal{F} = \{f_e\}_{e \in M}$ be a weakly-represented class of partial functions. Without loss of generality, assume that $\mathcal{F}$ is universal; that is, if there is a total function $f \leq_{\mathrm{T}} \mathcal{F}$, then $f \in \mathcal{F}$. (In particular, we can interleave a weak representation of a universal family of partial $\mathcal{F}$-computable functions much as we did with our universal diagonal in our proof of Theorem 3.5.6, ensuring that $\mathcal{F}$ satisfies **DNRW** if this broader class does.)

By **ID0**, there is a set A such that every injective $f \in \mathcal{F}$ samples A with density 0. By $\mathbf{RCA}_0$, there exists a function $f \leq_{\mathrm{T}} A$ with $f(n) = A\upharpoonright p_A(n!)$.

If there is some $N \in M$ such that $f(n) \neq f_n(n)$ for all $n > N$, then there must exist $\widehat{f}$ (a finite variation on f) such that $\widehat{f}(n) \neq f_n(n)$ for all $n \in M$; therefore, $\mathcal{F}$ satisfies **DNRW**.

Otherwise, we define

$$p(k) \coloneqq (\mu s)(\exists n \geq k)[f_n(n)[s] \downarrow = f(n)].$$

Such a function exists, since in this case p is total and computable from f.

Suppose there is some total $q \in \mathcal{F}$ such that $q(k) \geq p(k)$ for an unbounded set of $k \in M$; without loss of generality, we assume q is strictly increasing. We define a new function $h(n) = f_n(n)[q(n)]$ if this converges, and otherwise arbitrarily assign $h(n) = 0$. Of course, $h \leq_{\mathrm{T}} \mathcal{F}$, since $\mathcal{F}$ suffices to determine whether $f_n(n)$ converges by step $q(n)$.

Taking any k with $q(k) \geq p(k)$, we know (by the definition of $p(k)$) that there is some

$n \geq k$ such that $f_n(n)[p(k)] \downarrow f(n)$; therefore,

$$f(n) = f_n(n)[p(k)] = f_n(n)[q(k)] = f_n(n)[q(n)] = h(n).$$

Thus, since $q(k) \geq p(k)$ for an unbounded set of k's, there is also an unbounded set of n's such that $h(n) = f(n)$. We use this to construct a total injection $g \leq_{\mathrm{T}} h$ (and therefore $\leq_{\mathrm{T}} \mathcal{F}$) as follows:

Taking each $j \in M$ in increasing order, we find n such that $(n-1)! \leq j < n!$. Interpreting $h(n)$ as a binary string of length $n!$, let $(h(n))_j$ denote the position of the j-th 1 in this string if it exists; otherwise, take $(h(n))_j = 0$. We then set $g(j) = (h(n))_j$ if there is no $i < j$ with $g(i) = (h(n))_j$; otherwise, we set $g(j)$ to be the least number in M greater than $g(i)$ for all $i < j$.

Consider any n with $h(n) = f(n)$. We have $(h(n))_j = (f(n))_j$ for all $j < n!$, so for all $j \in [(n-1)!, n!)$, the j-th smallest element of A is equal to $g(i)$ for some $i \leq j$. Thus, $g([0, n!))$ contains at least $n! - (n-1)!$ elements of A, and therefore,

$$\rho_{n!}(g^{-1}(A)) \geq \frac{n! - (n-1)!}{n!} = 1 - \frac{1}{n}.$$

Since this holds for an unbounded set of n's, we conclude that A has upper density 1 under the sampling $g \leq_{\mathrm{T}} \mathcal{F}$.

However, since $\mathcal{F}$ is universal, this shows that A has upper density 1 for some $g \in \mathcal{F}$, contradicting our assumption every injective $f \in \mathcal{F}$ samples A with density 0. Therefore, it must be that for each total $q \in \mathcal{F}$, there is some $b \in M$ such that $q(k) < p(k)$ for all $k > b$; in other words, p is a dominating function for the total functions in $\mathcal{F}$, and so $\mathcal{F}$ satisfies **DOM**.

Thus, we see that every weakly-represented class of functions in $\mathcal{M}$ satisfies either **DNRW** or **DOM**. □

Theorem 3.5.10. $\mathbf{RCA}_0 + (\mathbf{DNRW} \vee \mathbf{DOM}) \models \mathbf{ID0}$.

Proof. Consider some weakly-represented family of (partial) functions $\mathcal{F} = \{f_e\}_{e \in M}$; without loss of generality, we assume $\mathcal{F}$ is universal. By **DNRW** $\vee$ **DOM**, there exists either a function F dominating all total functions in $\mathcal{F}$ or a function f with $f(n) \neq f_n(n)$ for all $n \in M$.

Suppose that there is a function F dominating all total functions in $\mathcal{F}$; without loss of generality, we may assume F to be strictly increasing. We define I to be the image of F:

$$I = \{n \in M : (\exists s)[f(s) = n]\}.$$

Suppose there is some total injective $f_e \in \mathcal{F}$ sampling I with positive upper density. We then choose some $q \in M$ with $\overline{\rho}(f_e^{-1}(S)) > \frac{1}{q}$, and define $h(n) = 1 + \max_{s \leq (n+1)q} f_e(s)$.

By our choice of f_e, there is an unbounded set of s's such that $f_e([0, s))$ contains at least $\frac{s}{q}$ values of F, and thus includes $F(m)$ for some $m \geq \frac{s}{q} - 1$. Therefore, there is an unbounded set of n's such that $g([0, (n+1)q))$ contains $F(n)$.

For each such n, we have $h(n) > F(n)$. Since F is a dominating function for all functions in $\mathcal{F}$, this implies that $h(n) \notin \mathcal{F}$. However, $h \leq_{\mathrm{T}} g \in \mathcal{F}$, contradicting our assumption that $\mathcal{F}$ is universal. Therefore, if every total function in $\mathcal{F}$ is dominated by F, every total injection in $\mathcal{F}$ samples I with density 0.

On the other hand, suppose instead that there is a function g with $g(n) \neq f_n(n)$ for all $n \in M$. Fix a universal-for-$\mathcal{F}$ machine U, and let Ψ be the computable functional such that

$$\Psi^X(x) = (\mu y)[\langle x, y \rangle \in X]$$

if for all $x' \leq x$, there exists some y' such that $\langle x', y' \rangle \in X$; otherwise, $\Psi^X(x) \uparrow$.

We then define $e(\sigma)$ (for any M-finite binary string σ) such that $f_{e(\sigma)} = \Psi^{U(\sigma)}$ (possible, since $\mathcal{F}$ is universal), and let

$$p(n) = 1 + \max_{|\sigma| < 5 \log n} \langle e(\sigma), g(e(\sigma)) \rangle.$$

Since $p \leq_{\mathrm{T}} g$, p exists in $\mathcal{M}$.

Using this, we define the set $P = \{A\upharpoonright p(n)\}_{n\in M}$, where $A = G(g)$; we will show that every total injective $f_e \in \mathcal{F}$ samples P with density 0.

Suppose, for the sake of contradiction, that there is some s such that f_e is a total injection sampling P with positive upper density; specifically, take this upper density to be greater than $\frac{1}{q}$. We define h so that $h(n)$ codes the M-finite set $D_{h(n)} = f_e(\left[0, n^2\right))$; clearly, $h \leq_{\mathrm{T}} f_e \leq_{\mathrm{T}} \mathcal{F}$.

By assumption, there is an unbounded set of s's such that $f_e([0, s))$ contains at least $\frac{s}{q}$ elements of P, including $(A\upharpoonright p(m))$ for some $m \geq \frac{s}{q} - 1$. Therefore, there is an unbounded set of n's such that $f_e([0, (n+1)q))$ contains $(A\upharpoonright p(n))$. Since $n^2 \geq (n+1)q$ for all sufficiently large n, there is an unbounded set of n's such that $(A\upharpoonright p(n)) \in D_{h(n)}$.

Take $k(n)$ to be the 2-to-1 prefix-free binary coding of $n \in M$ (with end symbol) in $2\log n + 2$ bits. For $x < n^2$, let $c_n(x)$ be the standard binary coding of x in $2\log n$ bits. Since U is a universal machine for $\mathcal{F}$ and $h \leq_{\mathrm{T}} \mathcal{F}$, there is a string σ such that, for $x < n^2$, $U(\sigma^\frown k(n)^\frown c_n(x))$ is the x-th element of $D_{h(n)}$.

For all n such that $(A\upharpoonright p(n)) \in D_{h(n)}$, we define $\sigma_n = \sigma^\frown k(n)^\frown c_n(x)$, where x is the index of $(A\upharpoonright p(n))$ in $D_{h(n)}$. By construction, $|\sigma_n| = 4\log n + |\sigma| + 2$, so for all sufficiently large n, we have $|\sigma_n| < 5\log n$.

Choose some N such that $(A\upharpoonright p(N)) \in D_{h(N)}$ and $|\sigma_N| < 5\log N$. By the definition of $e(\sigma)$, we have that

$$f_{e(\sigma_N)}(e(\sigma_N)) = \Psi^{U(\sigma_N)}(e(\sigma_N)) = \Psi^{A\upharpoonright p(N)}(e(\sigma_N)).$$

However, $\Psi^A(e(\sigma_N)) = g(e(\sigma_N))$, by our choice of A and Ψ. Since we assumed $g(e) \neq f_e(e)$ for any $e \in M$ where $f_e(e)$ converges, we must have $\Psi^A(e(\sigma_N)) \neq f_{e(\sigma_N)}(e(\sigma_N))$. Therefore,

$$\Psi^{A\upharpoonright p(N)}(e(\sigma_N)) \neq \Psi^A(e(\sigma_N)) = g(e(\sigma_N)).$$

In other words, $p(N) \leq \langle e(\sigma_N), g(e(\sigma_N))\rangle$. Since $|\sigma_N| < 5 \log n$, this contradicts our definition of p; therefore, no total injection in $\mathcal{F}$ samples P with positive upper density. □

Combining these, we obtain our desired result:

Corollary 3.5.11. ID0 *is equivalent to* **DNRW** $\vee$ **DOM** *over* $\mathbf{RCA}_0$.

Zhang and Stephan's principle **AVOID** is, of course, also relevant to our discussion, as it directly represents the existence of a function that is not weakly "computably" traceable.

Statement 3.5.12 (**AVOID**)**.** For every weakly-represented family of total functions $\mathcal{F}$, there is a function g such that for each $f \in \mathcal{F}$, the set $\{x \in M : f(x) = g(x)\}$ is bounded.

It is not difficult to see that our proof of Theorem 3.4.2 is valid in $\mathbf{RCA}_0$, directly showing that **ID0** implies **AVOID** over $\mathbf{RCA}_0$. To adapt our proof of Theorem 3.4.4 to show that **AVOID** implies **ID0**, however, we must again incorporate certain aspects of the equivalence from Kjos-Hanssen, Merkle, and Stephan [15]. Using much of the same approach as in our proof of Theorem 3.5.10 above, it is possible to adapt their proof of this equivalence to show that **AVOID** is equivalent to **DNRW** $\vee$ **DOM** over $\mathbf{RCA}_0$, as anticipated; we omit the details here. Combining these results with Corollary 3.5.11, we see that:

Theorem 3.5.13. *The following are equivalent over* $\mathbf{RCA}_0$*:*

- **ID0***,*
- **AVOID***, and*
- **DNRW** $\vee$ **DOM***.*

3.6 Intrinsic density and randomness

Let us move from the extremes of density (density 0 or 1) to the intermediate densities, as exemplified by density $\frac{1}{2}$.

The notion of "density $\frac{1}{2}$" is easily recognized as the Law of Large Numbers, as applied to a sequence of flips of a fair coin. We might hope that having density $\frac{1}{2}$ would be in some way related to a randomness-theoretic property, and stochasticity is the obvious candidate. This follows von Mises [28] in establishing the existence of limiting frequencies as the key property of a random sequence and, more specifically, the preservation of limiting frequencies under place-selection rules that determine the next bit sampled based only on the values previously sampled. If $\mathcal{C}$ is such a class of selection rules, we say that a sequence S is $\mathcal{C}$-stochastic if no selection rule in $\mathcal{C}$ can select a biased (non-density-$\frac{1}{2}$) subsequence from S. We say that a selection rule is monotonic if the places it selects are always in increasing order, and oblivious if the places it selects are independent of S, the sequence subject to the selection rule.

There are several standard notions of stochasticity that will be useful to keep in mind. Church-stochastic sequences are stochastic under computable monotonic selection rules, whereas von Mises-Wald-Church-stochastic sequences are stochastic under partial computable monotonic selection rules. By this definition, sets with density $\frac{1}{2}$ might be termed "trivially stochastic"; that is to say, they are unbiased under the single selection rule that selects all positions in order. However, this is rarely considered, as stochasticity is generally taken to require selection of proper subsequences.

Passing to intrinsic density $\frac{1}{2}$, we find something more practical: stochasticity under the class of all computable permutations, represented as oblivious selection rules. In fact, this is the class of non-monotonic oblivious selection rules that must eventually select every position. The corresponding notion of randomness, that no computable martingale succeeds on the sequence of bits selected by such a rule, is permutation randomness as defined by Miller and Nies [19]; intrinsic density $\frac{1}{2}$ is thus the natural notion of permutation stochasticity.

As mentioned above, stochasticity is generally taken to require selection of proper subsequences to preserve density $\frac{1}{2}$; this would seem to be an obstacle to considering intrinsic density $\frac{1}{2}$ as a valid notion of stochasticity. Fortunately, permutation stochasticity in fact ensures that many proper subsequences are also unbiased, including all computably-sampled

subsequences. We can be fully precise about this with one more definition in hand, and a combinatorial lemma to relate it to our previous work.

Definition 3.6.1. Given a total computable injection p and an infinite binary sequence $X(n)$, we say that the *subsequence of X sampled by p* is

$$p^{-1}(S) = \{X(p(n))\}_{n\in\omega}.$$

Abusing notation as noted in the introduction, we can apply this directly to any set $S \subseteq \omega$. In set notation, this gives

$$p^{-1}(S) = \{n \in \omega : p(n) \in S\}.$$

However, even though this new method of sampling generalizes our previous method of considering sets under computable permutations of ω, it has no additional power as far as density is concerned.

Lemma 3.6.2. *Given any total computable injection p, there is a computable permutation π such that, for any set S, $\pi^{-1}(S)$ has upper and lower density equal to those of $p^{-1}(S)$.*

Proof. Given a total computable injection p, we define a computable permutation π by assigning values $\pi(j)$ in increasing order of j. If j is a non-square integer, and $p(j)$ has not already been assigned to $\pi(i)$ for some $i < j$, define $\pi(j) = p(j)$. Otherwise, define $\pi(j)$ to be the least value not assigned to any $\pi(i)$ with $i < j$.

The sizes of $\pi([0, n)) \cap S$ and $p([0, n)) \cap S$ differ by at most $\lceil\sqrt{n}\rceil$. Thus, $\rho_n(\pi^{-1}(S))$ differs from $\rho_n(p^{-1}(S))$ by less than $\frac{2}{\sqrt{n}}$. Therefore,

$$\limsup_{n\to\infty} \rho_n(\pi^{-1}(S)) = \limsup_{n\to\infty} \rho_n(p^{-1}(S)),$$

and

$$\liminf_{n\to\infty} \rho_n(\pi^{-1}(S)) = \liminf_{n\to\infty} \rho_n(p^{-1}(S)).$$

□

From this minor lemma, we note that in fact, any set with intrinsic density has constant density not only under all computable permutations of ω, but also under all computable "samplings" of ω. To be more precise:

Corollary 3.6.3. *A set A has intrinsic density d iff $\rho(p^{-1}(A)) = d$ for every total computable injection p.*

Proof. The reverse direction is obvious by definition, since computable permutations of ω are also total computable injections.

The forward direction is, at this point, also quite straightforward. Fix a total computable injection p. By Lemma 3.6.2, there is a computable permutation π such that $\pi^{-1}(S)$ has the same upper and lower densities as $p^{-1}(S)$ for any set S, and in particular for A. Since A has intrinsic density d, we know that $\rho(\pi^{-1}(A)) = d$, and so that $\rho(p^{-1}(A)) = d$. □

This corollary reveals that intrinsic density $\frac{1}{2}$ coincides with another form of stochasticity: stochasticity under all computable injections, or equivalently the class of all oblivious non-monotonic selection rules. The corresponding notion of randomness is injection randomness, also as defined by Miller and Nies [19]. Thus, we see that:

Corollary 3.6.4. *Permutation stochasticity and injection stochasticity coincide, and are both equivalent to intrinsic density $\frac{1}{2}$.*

Considering this interpretation of intermediate intrinsic densities (strictly between 0 and 1) as a form of stochasticity, we find that intrinsic density provides an interesting link between the immunity properties and randomness-theoretic ideas. As discussed above, intrinsic density 0 is an immunity-type property, and so intrinsic density 1 is a form of co-immunity (or, as it is called for c.e. sets, simplicity). Thus, intrinsic density illustrates the relations between immunity, randomness, and simplicity, and provides a continuum of intermediate concepts, all of which follow in the spirit of stochasticity as established by von Mises. This

calls our attention to the fact that all of these properties are, in essence, descriptions of unpredictability: a set is immune if it is sufficiently difficult for a computable enumeration to stay within the set, co-immune if it is difficult to avoid the set, and stochastic if it is difficult to achieve any sort of persistent pattern of biased intersection with the set or its complement.

Of course, all of this relies fundamentally on our use of *intrinsic* density. Considering asymptotic density alone, we find no useful connection to computability or randomness. A set with density 0 need not be immune in any useful sense, as is made clear by considering the computable set of perfect squares. Taking the complement, we obtain a set with density 1 that is trivial to avoid. Moreover, density $\frac{1}{2}$ is a poor notion of randomness, as recognized by and before von Mises, carrying no real implications for unpredictability; for instance, the set of even numbers is "stochastic" in this sense, and yet is trivially predictable.

CHAPTER 4

INTRINSIC ASYMPTOTIC COMPUTATION

4.1 Introduction

Having found promising results by imposing computable invariance on asymptotic density, we end by returning to the motivating problem with which we began Chapter 3: strengthening Jockusch and Schupp's generic-case computability to obtain similar invariance.

In practical terms, the weakness of asymptotic computability was shown by Hamkins and Miasnikov [8], who demonstrated that, in several reasonable codings, the halting problem is in fact decidable on a computable set of asymptotic density 1, due to the density of trivially non-halting programs. This suggests that we should strengthen asymptotic computability, to avoid rendering the halting problem "decidable" for trivial reasons.

On the other hand, Rybalov [24] has shown that if we insist on convergence on a set with density exponentially approaching 1 (also known as strong generic-case computability), then the halting problem is instead undecidable. This in turn points out the fragility of our definitions, as we can see that certain results may depend on the rate of convergence of a set's asymptotic density rather than its density alone.

Furthermore, Corollary 3.1.2 has a somewhat unfortunate consequence for generic-case computability. For any problem, if there is an algorithm that converges on an infinite set of inputs, that algorithm becomes a generic-case solution for the problem under some alternate coding of the input. After all, the domain of the algorithm is necessarily c.e.; there is therefore some coding of the underlying problem (corresponding to a permutation of ω) under which the algorithm converges on a set of density 1. In other words, most natural problems have generic-case computable solutions (as defined by Jockusch and Schupp [12]) under some computable permutation. This gives us another reason to use a stricter notion of asymptotic computability.

In Section 4.2, we propose four such definitions for each notion of asymptotic compu-

tation, varying in degree of uniformity. As we then show in Section 4.3, most of these are strictly weaker than ordinary Turing computability (with only one known equivalent to ordinary computability), but even the weakest of our notions does not consider the halting problem (or, in fact, any nontrivial index set) to be computable.

4.2 Definitions

Returning to the original definition of generic-case complexity for group-theoretic problems, from Kapovich, Myasnikov, Schupp, and Shpilrain [14], we note that the authors defined a problem in a finitely generated group to have generic-case complexity $\mathcal{C}$ if and only if this complexity is independent of the choice of generating set. They specifically state that, though the worst-case complexity for most group-theoretic problems does not depend on one's choice of generating set, there is no reason to assume that this should also hold for generic-case complexity. As this choice directly corresponds to a coding of the input to the generic-case algorithm, a natural translation would require that our set be generic-case computable under every computable permutation of ω. Equivalently, by the Myhill Isomorphism Theorem, A should not be considered generic-case decidable unless all of the 1-equivalent sets are as well. Fortunately, this coincides with the standard idea that most computability-theoretic definitions are (or "should be") invariant under computable permutation.

We will call this new notion intrinsic generic-case computability, as it must be preserved under computable permutations of ω. Below, we propose four definitions, varying in degree of uniformity.

Our weakest candidate notion of intrinsic generic-case computability is the direct translation of the definition by Kapovich, Myasnikov, Schupp, and Shpilrain [14]:

Definition 4.2.1. A set A is *(weakly) intrinsically generic-case computable* iff $\pi(A)$ is generic-case computable for every computable permutation $\pi : \omega \to \omega$.

Note that we place no requirements on the relationships between the generic-case de-

scriptions for each such image $\pi(A)$; the algorithms may be essentially unrelated.

Insisting on a bare minimum of uniformity, we obtain our next candidate definition:

Definition 4.2.2. A set A is *(uniformly) intrinsically generic-case computable* iff there is a uniformly computable family of functions f_e such that, if φ_e is a computable permutation, f_e is a generic-case description of $\varphi_e(A)$; that is, f_e has density-1 domain and wherever $f_e(n)$ converges, it converges to $(\varphi_e(A))(n)$.

On the other hand, allowing our description to require an index may weaken our notion of uniformity; after all, this means that our description f cannot be given only a black-box oracle specifying the computable permutation, but actually requires knowledge of *how* the permutation can be computed — and in particular may depend on the specific program provided to compute the permutation.

Requiring the description to work with only an oracle might seem a trivial variation, but significant differences have been observed in analogous situations; specifically, in computable model theory, the index-based definition of uniform computable categoricity has been shown to be strictly weaker (and less natural) than the definition providing only an oracle. [4] (In general, any oracle-based definition must be at least as strong as the corresponding index-based definition, since it is well-established that there is a Turing-machine procedure allowing us to convert an index into an effective oracle.) We therefore include this option in our list of candidate notions. In this case, we would say that:

Definition 4.2.3. A set A is *(oracle) intrinsically generic-case computable* iff there is a Turing functional Φ^X such that, for any computable permutation π (represented as a set of pairs), Φ^π is a generic-case description of $\pi(A)$.

Finally, we might insist on complete uniformity, and require that a single algorithm provide a description of A on a set that has density 1 under all computable permutations; in other words, that the algorithm converge on a set of *intrinsic* density 1.

Definition 4.2.4. A set A is *(strongly) intrinsically generic-case computable* iff it has a description φ_e that converges on a set of intrinsic density 1. (Equivalently, $\varphi_e \circ \pi^{-1}$ is a generic-case description of $\pi(A)$ for all computable permutations π.)

More work will be required to distinguish these definitions of intrinsic generic-case computability, and some of them may prove to be equivalent. At this point, though, there are no reasons to presume any equivalences. The author personally expects that the uniform and strong definitions of intrinsic generic-case computability will be the most useful of these four.

Of course, similar notions exist for every form of intrinsic asymptotic computability. We will use these in the next section without further comment, as the definitions are analogous.

4.3 Properties of intrinsic asymptotic computation

Since r-maximal sets are c.e. and have intrinsic density 1, any r-maximal set is in fact strongly intrinsically generic-case computable. This provides a convenient demonstration that even this strongest definition is weaker than ordinary computability.

Of course, similar definitions exist for every form of intrinsic asymptotic computability; however, our observation above on r-maximal sets is only relevant to intrinsic generic-case and dense computability. For coarse computability, we note that any infinite set with intrinsic density 0 is strongly intrinsically coarsely computable, so again we see that intrinsic coarse computation is weaker than ordinary computation. However, strong intrinsic *effective* dense computation would require the description to converge correctly on a computable set of intrinsic density 1; as the only such sets are co-finite, we see that this definition collapses to ordinary computability. We leave the question of *weak* intrinsic effective dense computation to a later paper.

On the other hand, even our weakest definition has a certain demonstrable strength. A set $S \subseteq \omega$ is said to be an *index set* if $S(e) = S(e')$ for all $e, e' \in \omega$ where e and e' are indices

for equivalent Turing machines. Rice's Theorem [23] states that the only computable index sets are $\emptyset$ and ω. We can easily extend this to intrinsic dense computability, showing that no non-trivial index set can be weakly intrinsically densely computable. Therefore, the halting problem is not intrinsically densely computable under any of these definitions.

Theorem 4.3.1. *Suppose $S \subseteq \omega$ is an index set (i.e., $S(e) = S(e')$ for all e, e' such that $\varphi_e = \varphi_{e'}$). $\pi(S)$ is densely computable for all computable permutations π iff S is computable, and thus iff $S = \emptyset$ or $S = \omega$.*

Sketch of Proof. The reverse implication is obvious; we will only consider the forward implication.

By the Padding Lemma for Turing machines [25], for any e, we can computably enumerate a set $\{x_{e,0}, x_{e,1}, \ldots\}$ such that $\varphi_e = \varphi_{x_{e,i}}$ for all i. We can therefore build a computable injection $i : \omega \to \omega$ so that for all n, if $n \in [e!, (e+1)!)$, then $i(n) = x_{e,k}$, where k is the least j such that $i(m) \neq x_{e,j}$ for all $m < n$.

Suppose $i^{-1}(S)$ is densely computable, with dense description Ψ. There must be some E such that, for all $e > E$, $\Psi(n)$ converges to $i^{-1}(S)(n)$ for more than three-quarters of all $n \in [e!, (e+1)!)$; otherwise, Ψ could not converge to $i^{-1}(S)$ on a set of density 1. We can then determine whether $e \in S$ for all $e > E$; we simply wait until $\Psi(n)$ converges for at least three-quarters of $n \in [e!, (e+1)!)$ and then take the majority vote. Since this allows us to compute $S(e)$ for all but finitely many indices, S must be computable; therefore, by Rice's Theorem, $S = \emptyset$ or $S = \omega$. □

Corollary 4.3.2. *The halting problem is not (weakly) intrinsically densely computable.*

Proof. The halting problem is 1-equivalent to a non-computable index set (e.g., the set of all programs that halt for at least one input, $\{e \mid (\exists x)[\varphi_e(x)\downarrow]\}$). By the Myhill Isomorphism Theorem [20], this means that its image under some computable permutation is a non-computable index set. Composing this with the permutation from the proof of Theorem 4.3.1, we obtain a computable permutation under which the image of the halting problem is not

densely computable. Thus, the halting problem is not even weakly intrinsically densely computable. □

REFERENCES

[1] Laurent Bienvenu, Adam R. Day, and Rupert Hölzl. From bi-immunity to absolute undecidability. *The Journal of Symbolic Logic*, 78(4):1218–1228, 2013. ISSN 0022-4812. doi: 10.2178/jsl/1058448453.

[2] S. B. Cooper. Jump equivalence of the Δ_2^0 hyperhyperimmune sets. *The Journal of Symbolic Logic*, 37(3):598–600, 1972. ISSN 0022-4812. doi: 10.2307/2272750.

[3] Rodney G. Downey and Denis R. Hirschfeldt. *Algorithmic Randomness and Complexity*. Theory and Applications of Computability. Springer, 2010.

[4] Rodney G. Downey, Denis R. Hirschfeldt, and Bakhadyr Khoussainov. Uniformity in computable structure theory. *Algebra and Logic*, 42(5):318–332, 2003. ISSN 0002-5232. doi: 10.1023/A:1025971406116.

[5] Rodney G. Downey, Carl G. Jockusch, Jr., and Paul E. Schupp. Asymptotic density and computably enumerable sets. *Journal of Mathematical Logic*, 13(02), 2013. ISSN 0219-0613. doi: 10.1142/S0219061313500050.

[6] Rodney G. Downey, Carl G. Jockusch, Jr., Timothy H. McNicholl, and Paul E. Schupp. Asymptotic density and the Ershov hierarchy. *Mathematical Logic Quarterly*, to appear. ISSN 1521-3870. URL `http://arxiv.org/abs/1309.0137`.

[7] Yuri Gurevich. Average case completeness. *Journal of Computer and System Sciences*, 42(3):346–398, 1991. ISSN 0022-0000. doi: 10.1016/0022-0000(91)90007-R.

[8] Joel David Hamkins and Alexei Miasnikov. The halting problem is decidable on a set of asymptotic probability one. *Notre Dame Journal of Formal Logic*, 47(4):515–524, 2006. ISSN 0029-4527. doi: 10.1305/ndjfl/1168352664.

[9] Denis R. Hirschfeldt, Carl G. Jockusch, Jr., Rutger Kuyper, and Paul E. Schupp. Coarse reducibility and algorithmic randomness. To appear. Available at `http://www.math.uchicago.edu/~drh/Papers/coarsereducibility.html`.

[10] Gregory Igusa. Nonexistence of minimal pairs for generic computability. *The Journal of Symbolic Logic*, 78(2):511–522, 2013. ISSN 0022-4812. doi: 10.2178/jsl.7802090.

[11] Carl G. Jockusch, Jr. Upward closure and cohesive degrees. *Israel Journal of Mathematics*, 15(3):332–335, 1973. ISSN 0021-2172. doi: 10.1007/BF02787575.

[12] Carl G. Jockusch, Jr. and Paul E. Schupp. Generic computability, Turing degrees, and asymptotic density. *Journal of the London Mathematical Society*, 85(2):472–490, 2012. ISSN 0024-6107. doi: 10.1112/jlms/jdr051.

[13] Carl G. Jockusch, Jr. and Frank Stephan. A cohesive set which is not high. *Mathematical Logic Quarterly*, 39(1):515–530, 1993. ISSN 1521-3870. doi: 10.1002/malq.19930390153.

[14] Ilya Kapovich, Alexei Myasnikov, Paul E. Schupp, and Vladimir Shpilrain. Generic-case complexity, decision problems in group theory, and random walks. *Journal of Algebra*, 264(2):665–694, 2003. ISSN 0021-8693. doi: 10.1016/S0021-8693(03)00167-4.

[15] Bjørn Kjos-Hanssen, Wolfgang Merkle, and Frank Stephan. Kolmogorov complexity and the Recursion Theorem. *Transactions of the American Mathematical Society*, 363 (10):5465–5480, 2011. ISSN 0002-9947. doi: 10.1090/S0002-9947-2011-05306-7.

[16] Victor L. Klee, Jr. and George J. Minty. How good is the simplex algorithm? In Oved Shisha, editor, *Inequalities, III (Proc. Third Sympos., Univ. California, Los Angeles, Calif., Sept. 1–9, 1969)*, pages 159–175. Academic Press, New York, 1972.

[17] Leonid A. Levin. Average case complete problems. *SIAM Journal on Computing*, 15 (1):285–286, 1986. ISSN 0097-5397. doi: 10.1137/0215020.

[18] Donald A. Martin. Classes of recursively enumerable sets and degrees of unsolvability. *Mathematical Logic Quarterly*, 12(1):295–310, 1966. ISSN 1521-3870. doi: 10.1002/malq.19660120125.

[19] Joseph S. Miller and André Nies. Randomness and computability: Open questions. *The Bulletin of Symbolic Logic*, 12(3):390–410, 2006. ISSN 1079-8986. doi: 10.2178/bsl/1154698740.

[20] John R. Myhill, Sr. Creative sets. *Mathematical Logic Quarterly*, 1(2):97–108, 1955. ISSN 1521-3870. doi: 10.1002/malq.19550010205.

[21] André Nies. *Computability and Randomness*. Oxford Logic Guides. Oxford University Press, 2009.

[22] Emil L. Post. Recursively enumerable sets of positive integers and their decision problems. *Bulletin of the American Mathematical Society*, 50(5):284–316, 1944. ISSN 0002-9904. doi: 10.1090/S0002-9904-1944-08111-1.

[23] Henry G. Rice. Classes of recursively enumerable sets and their decision problems. *Transactions of the American Mathematical Society*, 74(2):358–366, 1953. ISSN 0002-9947. doi: 10.1090/S0002-9947-1953-0053041-6.

[24] Alexander Rybalov. Note: On the strongly generic undecidability of the halting problem. *Theoretical Computer Science*, 377(1-3):268–270, 2007. ISSN 0304-3975. doi: 10.1016/j.tcs.2007.02.010.

[25] Robert I. Soare. *Recursively Enumerable Sets and Degrees: A Study of Computable Functions and Computably Generated Sets*. Perspectives in Mathematical Logic. Springer-Verlag, 1987.

[26] Daniel A. Spielman and Shang-Hua Teng. Smoothed analysis of algorithms: Why the simplex algorithm usually takes polynomial time. *Journal of the ACM*, 51(3):385–463, 2004. ISSN 0004-5411. doi: 10.1145/990308.990310.

[27] Michiel van Lambalgen. The axiomatization of randomness. *The Journal of Symbolic Logic*, 55(3):1143–1167, 1990. ISSN 0022-4812. doi: 10.2307/2274480.

[28] Richard von Mises. On the foundations of probability and statistics. *Annals of Mathematical Statistics*, 12(2):191–205, 1941. ISSN 0003-4851. doi: 10.1214/aoms/1177731748.

[29] Jing Zhang and Frank Stephan. Weakly represented families in the context of reverse mathematics. Manuscript. Early version available at `http://www.comp.nus.edu.sg/ a0078129/FirstReport.pdf`.