Density and Computability

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Overview

Definition

Any set $S \subseteq \omega$ has *upper density*

$$\overline{\rho}(S) = \limsup_{n \to \infty} \frac{|S \upharpoonright n|}{n}$$

and *lower density*

$$\underline{\rho}(S) = \liminf_{n \to \infty} \frac{|S \upharpoonright n|}{n}.$$

If these coincide, S has *(asymptotic) density*

$$\rho(S) = \lim_{n \to \infty} \frac{|S \upharpoonright n|}{n}.$$

Virtues

- Intuitive
  - What fraction of ω is even? $\frac{1}{2}$.
  - What fraction of ω is divisible by n? $\frac{1}{n}$.
  - What fraction of ω is prime? 0.
- Content/pseudomeasure: like a measure, but finitely additive
  - Actually, slightly better...

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  - Actually, slightly better...

Theorem (Restricted countable additivity)

Let $\{S_j\}$ be a countable sequence of pairwise-disjoint subsets of ω with density. If $\lim_{n\to\infty} \overline{\rho}(\bigcup_{j=n}^{\infty} S_j) = 0$, then $\rho(\bigcup S_j) = \sum \rho(S_j)$.
[Jockusch and Schupp, 2012]

The Problem

Theorem (Density of computable sets [Downey, Jockusch, and Schupp, 2013])

For any left-(Σ^0_2, Π^0_2) *pair* (a, b) *with* $0 \leq a \leq b \leq 1$*, there is an (infinite co-infinite) computable set* A *with lower density* a *and upper density* b*.*

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For any left-(Σ^0_2, Π^0_2) pair (a, b) with $0 \leq a \leq b \leq 1$, there is an (infinite co-infinite) computable set A with lower density a and upper density b.

Corollary

For any infinite co-infinite computable A and any (Σ^0_2, Π^0_2) pair (a, b) with $0 \leq a \leq b \leq 1$, there is a computable permutation π such that $\pi(A)$ has lower density a and upper density b.

A set $S \subseteq \omega$ has *intrinsic density* $\boldsymbol{\rho}$ if it has density $\boldsymbol{\rho}$ under every computable permutation of ω; that is, for every computable permutation π,

$$\rho(\pi(S)) = \rho(S) = \boldsymbol{\rho}.$$

We define *intrinsic upper* and *lower density* analogously.

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More generally, a set S has *absolute upper density*

$$\overline{\boldsymbol{\rho}}(S) = \sup_{\pi} \overline{\rho}(\pi(S))$$

and *absolute lower density*

$$\underline{\boldsymbol{\rho}}(S) = \inf_{\pi} \underline{\rho}(\pi(S)).$$

Examples

Proposition

Every Schnorr random set has intrinsic density $\frac{1}{2}$.

Proof Sketch.

Schnorr randomness is computably invariant, and all Schnorr randoms obey the Law of Large Numbers. □

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Proposition (Jockusch)

Every r-cohesive (or even p-cohesive) set has intrinsic density 0.

Proof.

Any p-cohesive set is cofinitely contained in one equivalence class mod n; this holds true for all n. □

Sampling

Let $p : \mathbb{N} \to \mathbb{N}$ be a total injection.
If $\{a_n\}$ is a sequence, we say

$$p^{-1}(\{a_n\}) = \{a_{p(n)}\}$$

is the *subsequence sampled by p*.

The Sampling Lemma

Lemma ([Astor, 2015])

If p is a computable injection, there is a computable permutation π such that, for all X, $\pi^{-1}(X)$ and $p^{-1}(X)$ have the same upper and lower densities.

Construction.

$\pi(n) = p(n)$... unless:
n is a power of 2, or $\pi(j) = p(n)$ for some $j < n$.
In that case, let $\pi(n) = (\mu x)[x \notin \pi([0, n))]$. □

A Preliminary Equivalence

Definition

An *h-bounded weak trace* for f is a sequence of finite sets $D_{g(n)}$ with $|D_{g(n)}| \leq h(n)$, where $f(n) \in D_{g(n)}$ infinitely often.
A is *weakly computably traceable* if, for some computable h, every $f \leq_{\mathrm{T}} A$ has an h-bdd computable weak trace (i.e., $g \leq_{\mathrm{T}} \emptyset$).

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Theorem ([Kjos-Hanssen, Merkle, and Stephan, 2011])

The following are equivalent:

- *A has either DNC or high degree.*
- *A is not weakly computably traceable.*
- *$\exists f \leq_{\mathrm{T}} A$ s.t. if $h \leq_{\mathrm{T}} \emptyset$, $f(n) \neq h(n)$ for all suff. large n.*

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Lemma

If $G = \{\langle n, f(n)\rangle : n \in \mathbb{N}\}$ is the graph of f, and G does not have ID0, then f has a computable weak trace with bound $h(n) = n^2$.

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Proof.

If **a** is DNC or high, it is not WCT; **a** computes some f that has no computable weak trace with bound $h(n) = n^2$.
The graph of f has ID0. □

Lemma

If $G = \{\langle n, f(n)\rangle : n \in \mathbb{N}\}$ is the graph of f, and G does not have ID0, then f has a computable weak trace with bound $h(n) = n^2$.

Proof of Lemma.

For some permutation $\pi \leq_{\mathrm{T}} \emptyset$, $\pi^{-1}(G)$ has upper density $> \frac{1}{q}$.

Infinitely often, $|\pi^{-1}(G)\restriction s| > \frac{s}{q}$.
That is: i.o., $\pi([0,s))$ contains at least $\frac{s}{q}$ elements of G;
this includes $\langle m, f(m)\rangle$ for some $m > \frac{s}{q}$.
Therefore: i.o., $\langle m, f(m)\rangle \in \pi([0, mq))$.

Let $D_{g(n)} = \{y : \langle x, y\rangle \in \pi([0, nq))\}$ for $n > q$.
$|D_{g(n)}| \leq nq < n^2$, and $f(m) \in D_{g(m)}$ infinitely often. □

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If A has neither DNC nor high degree, then A has absolute upper density 1.

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Corollary

If A has neither DNC nor high degree, then A has absolute lower density 0.

Proof.

The complement of A has the same degree; apply the theorem. □

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Proof of Theorem.

Since A is WCT, for any $f \leq_{\mathrm{T}} A$, there is some $h \leq_{\mathrm{T}} \emptyset$ with $f(n) = h(n)$ i.o.

Let $A = \{a_1 < a_2 < \ldots\}$, and take $f(n) = \langle a_1, a_2, \ldots, a_{n!} \rangle$.
Let h be s.t. $h(n) = \langle a_1, a_2, \ldots, a_{n!} \rangle$ i.o.

Define g as follows.
If $(n-1)! \leq j < n!$, let $g(j) = h(n)_j$. Unless...
If $g(i) = h(n)_j$ for some $i < j$, let $g(j) = (\mu x)[x \notin g([0, j))]$.

g is injective.
When $h(n) = f(n)$, $|g([0, n!)) \cap A| \geq n! - (n-1)!$.
Therefore, i.o., $\rho_{n!}(g^{-1}(A)) \geq 1 - \frac{1}{n}$. □

Theorem

a *computes a set with intrinsic density 0 iff* **a** *is either DNC or high.*

- There are arithmetical infinite sets with ID0.
- If a set has ID0, all of its infinite subsets have ID0.
- Therefore: the Turing degrees of infinite sets with ID0 are closed upwards. [Jockusch (1970)]

Corollary

a *contains a set with intrinsic density 0 iff* **a** *is either DNC or high.*

Theorem

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Theorem

Any set with intrinsic density is Turing-equivalent to a set with ID0.

A Weaker Variant

What if we ask for the degrees containing a set with intrinsic *lower* density 0?

A Weaker Variant

Theorem

If A is a set, let $S = \{A \upharpoonright n : n \in \mathbb{N}\}$.
If S does not have ILD0, then A is computable.

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Theorem

If A is a set, let $S = \{A{\upharpoonright}n : n \in \mathbb{N}\}$.
If S does not have ILD0, then A is computable.

Proof.

There is a permutation $\pi \leq_{\mathrm{T}} \emptyset$, and some $q \in \mathbb{N}$, with $\rho_n(\pi^{-1}(S)) > \frac{1}{q}$ for all sufficiently large n.
For some m, if $n > m$, $|\pi([0, n)) \cap S| \geq \frac{n}{q}$.

Start with $T = 2^m$. Add σ to T if:

- all prefixes are in T, and
- $\pi([0, 2q|\sigma|))$ contains at least $|\sigma|$ extensions of σ.

T has width at most $2q$, and A is a path on T. □

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Schnorr randoms have ID1/2.
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The Turing degrees of sets with ID1/2 include all 1-random or high degrees, and include at most all DNC or high degrees.

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Open Question

What is the exact characterization of the ID1/2 degrees?

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The End

Questions?