Voting on $\mathbb{N}$:
Upper Cones for Asymptotic Computation

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Joint with Denis Hirschfeldt and Carl Jockusch.

Overview

1. Definitions and Relations
   - Definitions
   - Relations
   - Reducibilities
   - Minimal Pairs

2. Voting on the Natural Numbers
   - A Voting Lemma
   - Upper Cones for Asymptotic Computation
   - Minimal Pairs

The Idea

A total function f is *asymptotically computable* if it has a description that is correct on a set of density 1.

If g is a description of f, we say it is correct where $g(n)\downarrow = f(n)$. It may have two types of error:

- Omission: $g(n)\uparrow$
- Commission: $g(n)\downarrow \neq f(n)$

Definitions

g is a partial description of f if it has no errors of commission; that is, g is a partial function such that if $g(n)\downarrow$, then $g(n)\downarrow= f(n)$.
We say g is a *generic description* of f if its domain has density 1.
f is *generically computable* if it has a computable generic description.

g is a *coarse description* of f if it is asymptotically correct and has no errors of omission; that is, g is a total function, and $g(n) = f(n)$ on a set of density 1.
f is *coarsely computable* if it has a computable coarse description.

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g is a *dense description* of f if it is asymptotically correct; that is, g is a partial function such that $g(n)\downarrow= f(n)$ on a set of density 1.

Let g be a total $\omega \sqcup \{\square\}$-valued function. g is a strong partial description of f if $g(n) \in \{f(n), \square\}$. If $g^{-1}(\square)$ has density 0, then g is an *effective dense description* of f.

Wait - what was that last one?

Definition

Let g be a total $\omega \sqcup \{\Box\}$-valued function. g is a strong partial description of f if $g(n) \in \{f(n), \Box\}$. If $g^{-1}(\Box)$ has density 0, then g is an *effective dense description* of f.
f is *ed-computable* if there is a computable ed-description of f.

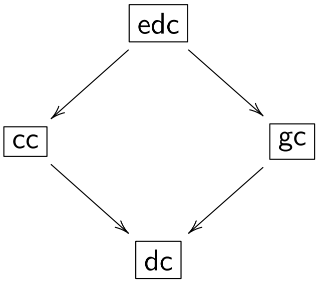
From this, we can obtain

$$g_g(n) = \begin{cases} g(n) & g(n) \in \omega, \\ \uparrow & g(n) = \Box, \end{cases}$$

and

$$g_c(n) = \begin{cases} g(n) & g(n) \in \omega, \\ 0 & g(n) = \Box. \end{cases}$$

Relations



Theorem ([Jockusch and Schupp, 2012])

There is a set that is coarsely computable, but not generically computable.

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There is a set that is generically computable, but not coarsely computable.

Reducibilities

None of these notions of relative asymptotic computation are transitive. (Oracles are full, not asymptotic.)

Switch to enumeration operators! $A \leq_c B$ if any coarse description of B computes a coarse description of A, and so on.

Each of these is transitive - so we get degree structures.

First (computability-inspired) question: are there minimal pairs?

Minimal Pairs

Theorem ([Igusa, 2013])

If X and Y are not generically comparable, then there is a set C generically computable from both X and Y that is not generically computable.
i.e., no minimal pairs for relative generic computation.

NOTE: It is still open whether generic reducibility has minimal pairs.

Theorem ([Hirschfeldt, Jockusch, Kuyper, and Schupp, to appear])

If X is not coarsely computable and Y is weakly 3-random relative to X, then X and Y are a minimal pair for relative coarse computation.

Towards Minimal Pairs

One approach - show that upper cones are small.
If $\{X : X$ asymptotically computes $A\}$ has measure 0, then a sufficiently random Y will compute nothing that X computes.

It suffices to show that $\Phi_A = \{X : \Phi^X$ is an asymptotic description of $A\}$ has measure 0 for each Turing functional Φ.

To do this — suppose not.
By Lebesgue density, some Φ_A has measure close to 1.
Start computing $\Phi^X(n)$ for all X; if a clear majority converge at n, then they must converge to $A(n)$, so the majority vote gives a correct answer.

But why should this happen at a density-1 set of n's?

Technical Lemma

Suppose uncountably many voters (each $X \in 2^\omega$) vote on countably many referenda (labeled by $n \in \omega$). Let $S_n =$ the class of voters supporting Proposition n, and let $S(X)$ be the set of referenda X supports (i.e., X's ballot).

Lemma

If $\mu(\{X : \rho(S(X)) = 1\}) > q$, then $\rho(\{n : \mu(S_n) \geq q\}) = 1$.

Think of it thia way: if each referendum needs measure-q support to pass, and more than measure-q voters supported most of the referenda, then most of the referenda will pass.

Upper Cones have Measure 0

Theorem

If A is not g.c., $\mu(\{X : A \text{ is generically } X\text{-computable}\}) = 0$.

Proof.

Suppose $A_\Phi = \{X \in 2^\omega : \Phi^X \text{ is a generic description of } A\}$ has $\mu > 0$.
By Lebesgue density, we may assume $\mu(A_\Phi) > \frac{3}{4}$.
Say X supports n if $\Phi^X(n) \downarrow = A(n)$. Clearly, $\mu(\{X : \rho(S(X)) = 1\}) > \frac{3}{4}$.

By the Lemma, $\rho(\{n : \mu(S_n) \geq \frac{3}{4}\}) = 1$... so for density-1 many n, there are at least measure-$\frac{3}{4}$ sets X with $\Phi^X(n) = A(n)$.

Define $f(n)$ by waiting to see $\Phi^X(n)$ converge on a class of measure at least $\frac{2}{3}$, then taking the majority-rule value.
f is a computable generic description of A. □

Upper Cones have Measure 0

Theorem

If A is not g.c., $\mu(\{X : A \text{ is generically } X\text{-computable}\}) = 0$.

Theorem ([Hirschfeldt, Jockusch, Kuyper, and Schupp, to appear])

If A is not c.c., $\mu(\{X : A \text{ is coarsely } X\text{-computable}\}) = 0$.

Theorem

If A is not d.c., $\mu(\{X : A \text{ is densely } X\text{-computable}\}) = 0$.

Theorem

If A is not e.d.c., $\mu(\{X : A \text{ is effectively densely } X\text{-computable}\}) = 0$.

Minimal Pairs for Dense Computation

Theorem

If Y is not densely computable, and X is weakly 4-random relative to Y, then X and Y are a minimal pair for dense computation.

Proof.

Suppose C is densely computable from both X and Y.
Fix $\{0,1\}$-valued dense descriptions Φ^X and Ψ^Y.

Let P be a set both low and PA over Y.
P computes a $\{0,1\}$-valued completion of Ψ^Y – a set D.

Φ^X is still a dense description of D.
Since P was low over Y, X is still weakly 4-random over P (and D).
But Φ_D is a measure-0 $\Pi_4^{0,D}$ set; D must be densely computable. □

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The End