Robust computation modulo "small" sets

Eric P. Astor

The University of Connecticut

eric.astor@uconn.edu

March 4, 2017

Overview

Motivation: Some hard problems are "usually" easy

Problem: what does "usually" mean?

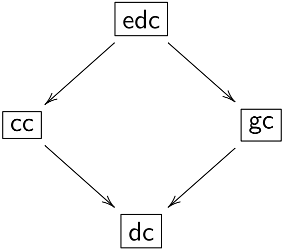
Defining "small" sets
- Candidates
- Intrinsic density
- ID0 and Immunity

Intrinsic computability

Computability Almost Everywhere

- Isn't computability supposed to study impossible problems?
- We've started realizing – for many of these, "almost every" case is easy.
  - Naïve simplex algorithm;
    high performance in practice, exponential time in theory.
  - Boolean satisfiability:
    prototypical NP-complete problem, now routinely solved.
  - Word problem for groups:
    most instances in many (most?) groups are quickly solved.
- One approach: work modulo a *vanishing fraction* of cases.
- In complexity: generic complexity
  [Kapovich, Myasnikov, Schupp, and Shpilrain, 2003]
- In computability:
  - Generic and coarse computability (Jockusch & Schupp)
  - Dense and effective dense computability (Astor, Hirschfeldt, & Jockusch)

Current Approaches



Theorem ([Jockusch and Schupp, 2012])

There is a set that is coarsely computable, but not generically computable.

Theorem ([Jockusch and Schupp, 2012])

There is a set that is generically computable, but not coarsely computable.

Asymptotic Density

Any set $S \subseteq \omega$ has *upper density*

$$\overline{\rho}(S) = \limsup_{n\to\infty} \frac{|S \restriction n|}{n}$$

and *lower density*

$$\underline{\rho}(S) = \liminf_{n\to\infty} \frac{|S \restriction n|}{n}.$$

If these coincide, S has *(asymptotic) density*

$$\rho(S) = \lim_{n\to\infty} \frac{|S \restriction n|}{n}.$$

Virtues

- Intuitive
  - What fraction of ω is even? $\frac{1}{2}$.
  - What fraction of ω is divisible by n? $\frac{1}{n}$.
  - What fraction of ω is prime? 0.
- Content/pseudomeasure: like a measure, but finitely additive
  - Actually, slightly better...

Virtues

- Intuitive
  - What fraction of ω is even? $\frac{1}{2}$.
  - What fraction of ω is divisible by n? $\frac{1}{n}$.
  - What fraction of ω is prime? 0.
- Content/pseudomeasure: like a measure, but finitely additive
  - Actually, slightly better...

Theorem (Restricted countable additivity)

Let $\{S_j\}$ be a countable sequence of pairwise-disjoint subsets of ω with density. If $\lim_{n\to\infty} \overline{\rho}(\bigcup_{j=n}^{\infty} S_j) = 0$, then $\rho(\bigcup S_j) = \sum \rho(S_j)$.
[Jockusch and Schupp, 2012]

Vices

- A density-0 set is "thin", but usually not immune.
  - Powers of 2, primes, etc.

Vices

- A density-0 set is "thin", but usually not immune.
  - Powers of 2, primes, etc.

Theorem (Density of computable sets [Downey, Jockusch, and Schupp, 2013])

For any infinite co-infinite computable A and any (Σ^0_2, Π^0_2) pair (a, b) with $0 \leq a \leq b \leq 1$, there is a computable permutation π such that $\pi(A)$ has lower density a and upper density b.

Vices

- A density-0 set is "thin", but usually not immune.
  - Powers of 2, primes, etc.

Theorem (Density of computable sets [Downey, Jockusch, and Schupp, 2013])

For any infinite co-infinite computable A and any (Σ^0_2, Π^0_2) pair (a, b) with $0 \leq a \leq b \leq 1$, there is a computable permutation π such that $\pi(A)$ has lower density a and upper density b.

Corollary

If A is infinite and not immune, there is a computable permutation π such that $\rho(\pi(A)) = 1$.

Vices

- A density-0 set is "thin", but usually not immune.
  - Powers of 2, primes, etc.

Theorem (Density of computable sets [Downey, Jockusch, and Schupp, 2013])

For any infinite co-infinite computable A and any (Σ^0_2, Π^0_2) pair (a, b) with $0 \leq a \leq b \leq 1$, there is a computable permutation π such that $\pi(A)$ has lower density a and upper density b.

Corollary

If A is infinite and not bi-immune, there is a computable permutation π such that $\pi(A)$ is effectively densely computable.

One Downside

- Hamkins and Miasnikov: The halting problem is decidable on a set of asymptotic probability one.

Theorem

The halting problem $K = \{e : \phi_e(0)\downarrow\}$ *is effectively densely computable...*

One Downside

- Hamkins and Miasnikov: The halting problem is decidable on a set of asymptotic probability one.

Theorem

The halting problem $K = \{e : \phi_e(0)\downarrow\}$ *is effectively densely computable... for a Turing machine with one-way infinite tape.*

- The reason: Most k-state programs either repeat a state or fall off the tape within k steps.

Extreme Options

- Dzhafarov & Igusa: Finite sets are small!
- If finite sets are small: mod-finite and cofinite reducibility
  - mod-finite: uniform Turing functional; if the oracle has finitely many errors, so does the output.
  - cofinite: uniform Turing functional; if the oracle has cofinite domain, so does the output.

Extreme Options

- Dzhafarov & Igusa: Finite sets are small!
- If finite sets are small: mod-finite and cofinite reducibility
  - mod-finite: uniform Turing functional; if the oracle has finitely many errors, so does the output.
  - cofinite: uniform Turing functional; if the oracle has cofinite domain, so does the output.
- Dzhafarov & Igusa: Alternatively, infinite sets are large!
  (so any non-cofinite set is small...)
- Infinite information reducibility
  - uniform Turing functional; if the oracle has infinite domain, so does the output.

Analyzing the Problem

- We don't care about solving all cases in order; we care about solving the $f(n)$-th case (f computable).

Analyzing the Problem

- We don't care about solving all cases in order; we care about solving the $f(n)$-th case (f computable).
- We care if $f^{-1}(B)$ is usually computable, not if B is!

Definition

A set $S \subseteq \omega$ has *intrinsic density* $\boldsymbol{\rho}$ if its computable preimages all have density $\boldsymbol{\rho}$; that is, for every computable injection f,

$$\rho(f^{-1}(S)) = \rho(S) = \boldsymbol{\rho}.$$

We define *intrinsic upper* and *lower density* analogously.

Definition

A set $S \subseteq \omega$ has *intrinsic density* $\boldsymbol{\rho}$ if its computable preimages all have density $\boldsymbol{\rho}$; that is, for every computable injection f,

$$\rho(f^{-1}(S)) = \rho(S) = \boldsymbol{\rho}.$$

We define *intrinsic upper* and *lower density* analogously.

More generally, a set S has *absolute upper density*

$$\overline{\boldsymbol{\rho}}(S) = \sup_f \overline{\rho}(f^{-1}(S))$$

and *absolute lower density*

$$\underline{\boldsymbol{\rho}}(S) = \inf_f \underline{\rho}(f^{-1}(S)).$$

A Note

Theorem (Astor)

For any injection $f \leq_{\mathrm{T}} \emptyset$, there is a permutation $\pi \leq_{\mathrm{T}} \emptyset$ such that $f^{-1}(S)$ and $\pi(S)$ have the same upper and lower densities for all S.

- Computable permutations suffice to define intrinsic density.

Finding Sets with ID0

Proposition (Jockusch)

Every r-cohesive set has intrinsic density 0.

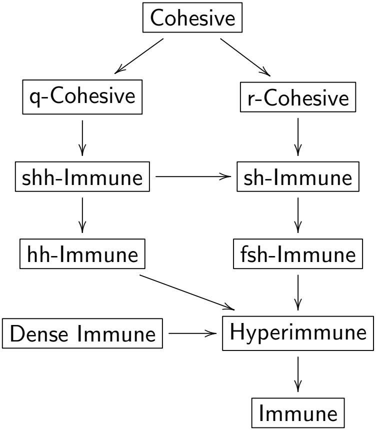
Proof: cofinitely contained in one equivalence class mod n, for any n. □

Proposition (Astor)

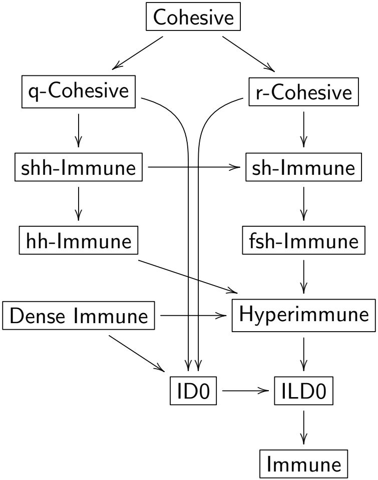
Every infinite set with intrinsic density 0 is immune.

Proof: Non-immune sets have density 1 under some permutation. □

The Classical Immunity Hierarchy



Now with Intrinsic Density 0



How Complex are ID0’s?

Theorem (Astor)

The Turing degrees of infinite sets with intrinsic density 0 are precisely the high or DNC degrees.

Intrinsic Generic Computability

Candidate definitions:

- Strong: algorithm correct for A when it converges; diverges only on a set with ID0.
- Oracle uniform: Turing functional Φ such that Φ^{π} is correct for $\pi(A)$, but diverges on a set of density 0.
- Uniform: Turing functional Φ such that if $\phi_e = \pi$, then Φ^e is correct for $\pi(A)$, but diverges on a set of density 0.
- Weak: $\pi(A)$ is generically computable for every permutation $\pi \leq_{\mathrm{T}} \emptyset$.

Theorem (Astor)

There is a permutation π such that, for any index set S, $\pi(S)$ is densely computable iff S is computable. Thus, no non-trivial index set is weakly intrinsically densely computable.

Sketch: Use the Padding Lemma; enumerate equivalent programs for every index and choose π to concentrate them.

Theorem (Astor)

There is a permutation π such that, for any index set S, $\pi(S)$ is densely computable iff S is computable. Thus, no non-trivial index set is weakly intrinsically densely computable.

Sketch: Use the Padding Lemma; enumerate equivalent programs for every index and choose π to concentrate them.

Corollary

However we define intrinsic dense computation, the halting problem is not i.d.c.

References

E. P. Astor, 2015.
Asymptotic density, immunity, and randomness.
Computability 4(2).

R. Downey, C. Jockusch, Jr., and P. Schupp, 2013.
Asymptotic density and computably enumerable sets
J. Math. Log. 13(2).

I. Kapovich, A. Myasnikov, P. Schupp, and V. Shpilrain, 2003.
Generic-case complexity, decision problems in group theory, and random walks
J. Algebra 264(2), 665–694.

C. Jockusch, Jr. and P. Schupp, 2012.
Generic computability, turing degrees, and asymptotic density
J. Lond. Math. Soc. 85(2), 472–490.

The End

Classical Immunity Properties

- immune – no c.e. subset
- dense immune – principal function dominates all computable functions

- array – uniform list of disjoint sets
- hyperimmune – avoids an element from each array of finite sets
- fsh-immune – avoids an element from each array of computable sets (all finite)
- sh-immune – avoids an element from each array of comp. sets (comp. union)
- hh-immune – avoids an element from each array of c.e. sets (all finite)
- shh-immune – avoids an element from each array of c.e. sets

- cohesive – infinite intersection with exactly one of A or $\overline{A}$ for all c.e. A
- r-cohesive – infinite intersection with exactly one of A or $\overline{A}$ for all computable A
- q-cohesive – finite union of cohesive sets