The uniform content of ADS

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Overview

Π^1_2 Problems

Principles of a particular form:

$$P: \ (\forall X)[\Phi(X) \to (\exists Y)[\Psi(X, Y)]],$$

with arithmetic formulas Φ and Ψ.

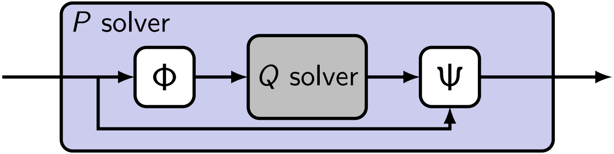
We say P is a *problem*.
X satisfying $\Phi(X)$ are *P-instances*.
Y is a *P-solution* to a P-instance X if $\Psi(X, Y)$.

Examples:

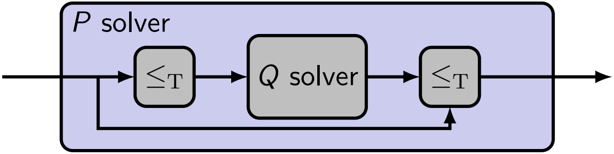
Ramsey's theorem (RT^n_k) — instances = colorings
Chain/antichain (CAC) — instances = partial orders
Ascending/descending seq. (ADS) — instances = linear orders

Reductions between Problems

Weihrauch reducibility:
$P \leq_W Q$ if we can uniformly convert a Q-solver into a P-solver.



Computable reduction:
$P \leq_c Q$ if P-instances are computably solvable using a Q-solver.



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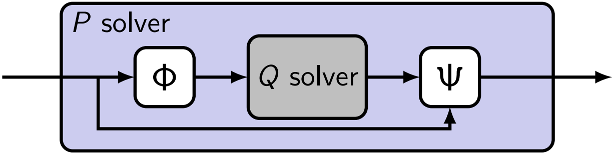
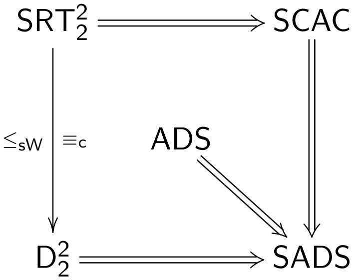


Example:
Given a linear order $\leq_L$, we define a coloring of pairs $(x < y)$;

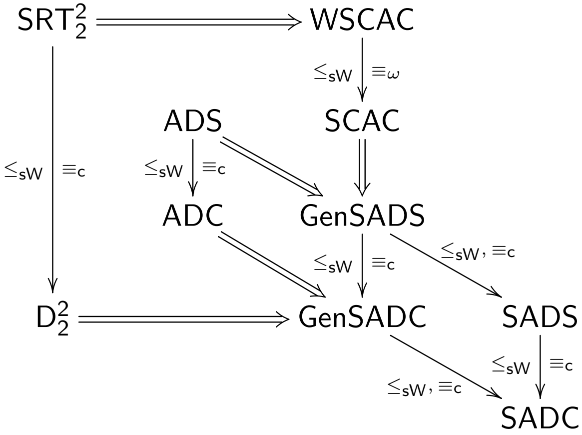
$$c(x, y) = \begin{cases} 0 & \text{if } y <_L x \\ 1 & \text{if } x <_L y \end{cases}$$

S is homogeneous for c iff S is a monotonic sequence for $\leq_L$.
Thus, $\mathrm{ADS} \leq_{sW} \mathrm{RT}^2_2$.

Known Relations



Fine Structure



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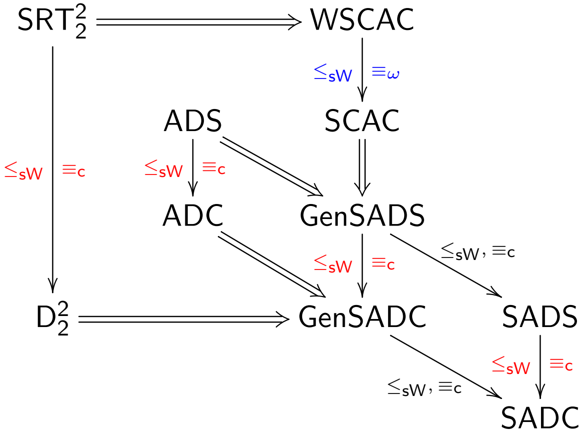


ADS and ADC

Consider a linear order $\leq_L$;

an *ascending (descending) chain* is an infinite set C, each $x \in C$ having only finitely many predecessors (successors) in C.

an *ascending sequence* is an infinite set S, where:
for all $x, y \in S$, we have $x \leq y$ iff $x \leq_L y$.

a *descending sequence* is an infinite set S, where:
for all $x, y \in S$, we have $x \leq y$ iff $y \leq_L x$.

ADC: Every inf. linear order $\leq_L$ has an infinite monotone chain.
ADS: Every inf. linear order $\leq_L$ has an infinite monotone sequence.

ADS vs. ADC

ADC: Every inf. linear order $\leq_L$ has an infinite monotone chain.
ADS: Every inf. linear order $\leq_L$ has an infinite monotone sequence.

Typically identified, since ADS $\equiv_c$ ADC.
ADS-instances are ADC-instances; only the solutions differ, subtly.

Monotonic sequences *are* chains.
Given a monotonic chain, we can extract a sequence.

- An ADS-instance is an ADC-instance.
- We have *two* functionals, and given an ADC-solution to L, *one* of them will produce an ADS-solution (a sequence).

This is *almost* uniform — one bit of non-uniform information.

Stable versions

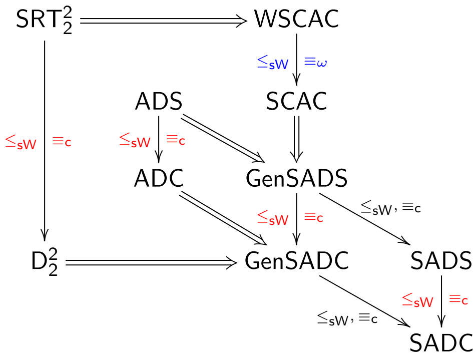
$$\begin{aligned}\text{We say } x \text{ is } \leq_O\text{-small if } & (\forall^\infty y)[x \leq_O y],\\ \leq_O\text{-large if } & (\forall^\infty y)[y \leq_O x],\\ \leq_O\text{-isolated if } & (\forall^\infty y)[x \perp_O y].\end{aligned}$$

An infinite partial order is *weakly stable* if all elements are small, large, or isolated, and *stable* if only one of small or large appears.

An infinite linear order is *stable* if all elements are small or large: $\omega + k$, $k + \omega^*$, or $\omega + \omega^*$.

- SADS/SADC: ADS/ADC for linear orders of type $\omega + \omega^*$.
- GenSADS/GenSADC: ADS/ADS for stable linear orders.

Fine Structure



New uniform results:
SADS $\not\leq_W$ ADC, SADS $\not\leq_W$ D$^2_2$, and GenSADC $\not\leq_W$ SADS.

One more: WSCAC $\not\leq_c$ SCAC.

Uniform results

New uniform results: SADS $\not\leq_{\mathrm{W}}$ ADC, SADS $\not\leq_{\mathrm{W}}$ D^2_2, ...

Key features of SADS:

- Solutions' types are locally detectable.
- With appropriate forcing, generic instances do not "self-solve".

Key features of ADC and D^2_2:

- Instances that do not "self-solve" have solutions of both types.
- Restrictions of instances ($Y \upharpoonright R$, R infinite) are instances; solutions to restrictions still solve the original problem.

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- ... and the interesting case of a Seetapun-style construction succeeds.

Proof structure

New uniform results: SADS $\not\leq_W$ ADC, SADS $\not\leq_W$ D^2_2, ...

Actual construction: augmented version of Seetapun and Slaman (originally used to separate ACA_0 from RT^2_2).

We look for ascending/descending "blobs" for Ψ:
a finite F is an *ascending blob* if

$$(\exists x < y)[(x <_L y) \wedge (x, y \in \Psi^{L \oplus F})].$$

If we find $F_0 < F_1 < F_2 < \ldots$ (all ascending or all descending), we build a *Seetapun tree* of "threads" in $\omega^{<\omega}$:

$\alpha \in T$ iff $\alpha(i) \in F_i$ and $\mathrm{range}(\alpha \restriction |\alpha| - 1)$ contains no blob.

Proof idea

New uniform results: SADS $\not\leq_W$ ADC, SADS $\not\leq_W$ D^2_2, ...

Three cases on each side (ascending/descending):

(i) Infinite sequence of blobs, finite Seetapun tree
(ii) Infinite sequence of blobs, infinite Seetapun tree
(iii) No infinite sequence of blobs

Case (i) is usually the interesting case in a Seetapun construction; here, it's standard.

In cases (ii) and (iii): If case (i) fails (say for ascending), then there is a subset containing no ascending blobs...

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Thank you!