Density, Intrinsic Density, and 'Usually Solvable' Problems

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Overview

Solvable Problems

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One can find its answer by some *reliable*, *systematic* method.

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Our answer: Turing's thesis.

A problem has a *computable* solution if we can give an algorithm that, given any instance of our problem, will always stop and output the correct answer.

'Usually Solvable' Problems

- First: not all problems are solvable.
  - Turing (1936): the halting problem for Turing machines.
  - SAT (boolean satisfiability): solvable, but NP-complete
  - Word problem for groups: equivalent to the halting problem.

'Usually Solvable' Problems

- First: not all problems are solvable.
  - Turing (1936): the halting problem for Turing machines.
  - SAT (boolean satisfiability): solvable, but NP-complete
  - Word problem for groups: equivalent to the halting problem.
- Except... we solve these problems all the time.
  - SAT solvers: applied to hardware verification!
  - Word problem: solvers used for proof automation, and in industry for equivalence of representations.

Definition

Any set $S \subseteq \omega$ has *upper density*

$$\overline{\rho}(S) = \limsup_{n\to\infty} \frac{|S \upharpoonright n|}{n}$$

and *lower density*

$$\underline{\rho}(S) = \liminf_{n\to\infty} \frac{|S \upharpoonright n|}{n}.$$

If these coincide, S has *(asymptotic) density*

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Theorem (Restricted countable additivity)

Let $\{S_j\}$ be a countable sequence of pairwise-disjoint subsets of ω with density. If $\lim_{n\to\infty} \overline{\rho}(\bigcup_{j=n}^{\infty} S_j) = 0$, then $\rho(\bigcup S_j) = \sum \rho(S_j)$.
[Jockusch and Schupp, 2012]

- Intuitive
  - What fraction of ω is even? $\frac{1}{2}$.
  - What fraction of ω is divisible by n? $\frac{1}{n}$.
  - What fraction of ω is prime? 0.

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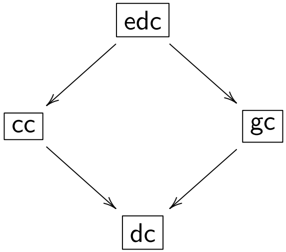
A problem is *generically computable* if there is an algorithm that never gives a wrong answer, and halts on a set of density 1.

A problem is *effectively densely computable* if there is an algorithm that always halts, gives the correct answer on a set of density 1, and otherwise outputs “?”.

Relations



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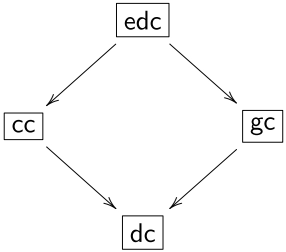

Theorem ([Jockusch and Schupp, 2012])

There is a set that is coarsely computable, but not generically computable.

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Theorem (Density of computable sets [Downey, Jockusch, and Schupp, 2013])

For any infinite co-infinite computable A and any (Σ^0_2, Π^0_2) pair (a, b) with $0 \leq a \leq b \leq 1$, there is a computable permutation π such that $\pi(A)$ has lower density a and upper density b.

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Corollary

If A is infinite and c.e., there is a computable permutation π such that $\rho(\pi(A)) = 1$.

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Corollary

If A is infinite and not bi-immune, there is a computable permutation π such that $\pi(A)$ is generically computable.

A set $S \subseteq \omega$ has *intrinsic density* $\boldsymbol{\rho}$ if it has density $\boldsymbol{\rho}$ under every computable permutation of ω; that is, for every computable permutation π,

$$\rho(\pi(S)) = \rho(S) = \boldsymbol{\rho}.$$

We define *intrinsic upper* and *lower density* analogously.

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More generally, a set S has *absolute upper density*

$$\overline{\boldsymbol{\rho}}(S) = \sup_{\pi} \overline{\rho}(\pi(S))$$

and *absolute lower density*

$$\underline{\boldsymbol{\rho}}(S) = \inf_{\pi} \underline{\rho}(\pi(S)).$$

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A is *permutation (injection) random* if it defeats every computable martingale selecting bits by a computable permutation (injection).

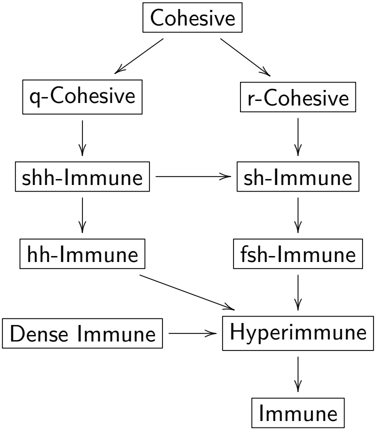
Theorem (Astor)

Permutation and injection stochasticity coincide, and are equivalent to intrinsic density $\frac{1}{2}$.

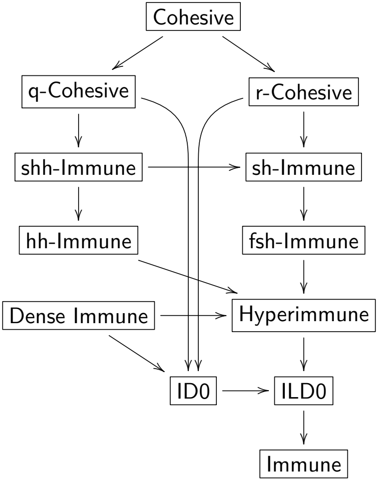
Unlike density, intrinsic density 0 (ID0) *is* an immunity property.
(Even ILD0 is strictly between hyperimmunity and immunity.)

In practical terms: weaker than dense immune, but more standard.

The Classical Immunity Hierarchy



Now with Intrinsic Density 0



New Results

Proposition (Jockusch)

Every r-cohesive set has intrinsic density 0.

Proof: cofinitely contained in one equivalence class mod n, for any n. □

Proposition (Astor)

Every dense immune set has intrinsic density 0.

Proof: combinatorial exercise with limits.

Proposition (Astor)

Every 1-random computes an infinite non-hyperimmune set with intrinsic density 0.

Proof: take the proof that 1-random sets are not hyperimmune, and use it to guide a progressive intersection of countably many relatively 1-random sets.

New Results

Theorem (Astor)

For all $\varepsilon > 0$, there exists a Δ^0_2 (s)hh-immune with upper density at least $1 - \varepsilon$.

Sketch of Proof.

Direct finite-injury construction below $\emptyset'$; make sure we only avoid "small" elements of each weak array, with small lower density.

The problem: $\emptyset'$ can't even approximate lower density for c.e. sets.

Instead, use $\emptyset'$ to approximate upper density for many disjoint c.e. sets at once, then use **that** to approximate when *other* disjoint c.e. sets must have small density. □

Note:

- Intrinsic density 0 is an immunity property.
- Intrinsic density $\frac{1}{2}$ is a form of stochasticity. (Same for any intrinsic density in $(0, 1)$.)
- Intrinsic density *connects* immunity and stochasticity

Something we forget:

- Immune ("thin"): hard to hit repeatedly
- Simple ("thick"): hard to avoid contact
- Stochastic: hard to achieve *any* structured pattern of intersection or non-intersection

Obviously related, and all about unpredictability.

Intrinsic Dense Computability

Candidate definitions:

- Weak: A is *weakly* i.d.c. iff $\pi(A)$ is densely computable for every computable permutation π.
- Uniform: A is *uniform* i.d.c. iff there is a uniform program that, provided an index for a permutation $\phi_e = \pi$, produces a dense computation of $\pi(A)$.
- Oracle uniform: A is *oracle uniform* i.d.c. iff there is a Turing functional Φ^X such that, for any computable permutation π, Φ^π is a dense computation of $\pi(A)$.
- Strong: A is *strongly* i.d.c. iff there is a computable function f such that $f(n)\downarrow= A(n)$ on a set of intrinsic density 1.

Theorem (Astor)

There is a permutation π such that, for any index set S, $\pi(S)$ is densely computable iff S is computable. Thus, $\pi(S)$ is not densely computable for any non-trivial index set S.

Sketch: Use the Padding Lemma; enumerate equivalent programs for every index and choose π to concentrate them.

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Corollary

Regardless of one's choice of definition of intrinsic dense computability, the halting problem is not i.d.c.

References

R. Downey, C. Jockusch, Jr., and P. Schupp, 2013.
Asymptotic density and computably enumerable sets
J. Math. Log. 13(2).

D. Hirschfeldt, C. Jockusch, Jr., R. Kuyper, and P. Schupp, 2015.
Coarse reducibility and algorithmic randomness
J. Symb. Log., to appear.

I. Kapovich, A. Myasnikov, P. Schupp, and V. Shpilrain, 2003.
Generic-case complexity, decision problems in group theory, and random walks
J. Algebra 264(2), 665–694.

B. Kjos-Hanssen, W. Merkle, and F. Stephan, 2011.
Kolmogorov complexity and the Recursion Theorem
Trans. Amer. Math. Soc. 363(10), 5465–5480.

C. Jockusch, Jr. and P. Schupp, 2012.
Generic computability, turing degrees, and asymptotic density
J. Lond. Math. Soc. 85(2), 472–490.

F. Stephan and Zhang J., preprint.
Weakly represented families in the context of reverse mathematics

The End

Classical Immunity Properties

- immune – no c.e. subset
- dense immune – principal function dominates all computable functions

- array – uniform list of disjoint sets
- hyperimmune – avoids an element from each array of finite sets
- fsh-immune – avoids an element from each array of computable sets (all finite)
- sh-immune – avoids an element from each array of comp. sets (comp. union)
- hh-immune – avoids an element from each array of c.e. sets (all finite)
- shh-immune – avoids an element from each array of c.e. sets

- cohesive – infinite intersection with exactly one of A or $\overline{A}$ for all c.e. A
- r-cohesive – infinite intersection with exactly one of A or $\overline{A}$ for all computable A
- q-cohesive – finite union of cohesive sets