Divisions in the Reverse Math Zoo, and the weakness of typicality

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Results joint with Bienvenu, Dzhafarov, Patey, Shafer, Solomon, and Westrick

Overview

What Zoo?

The Big Five systems have been essential to reverse mathematics, and significantly coincide with philosophically important divisions in proof techniques.

- RCA_0: constructive/computable mathematics
- WKL_0: compactness arguments
- ACA_0: "most down-to-earth constructions", PA, and predicativism

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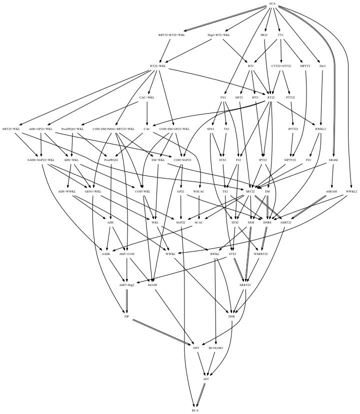
Conveniently, they all sit in a linear order.
But... There are exceptional theorems.

Oh, THAT Zoo

There are a *lot* of exceptional theorems.



A Shameless Plug

Before I go on: how did I make that diagram?

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The **RM Zoo**: `http://rmzoo.math.uconn.edu`

- Input: a bibliography, annotated with theorems proven.
- Output 1: all inferrable results (with justifications)
- Output 2: diagrams of the resulting Zoo, in adjustable detail

So What?

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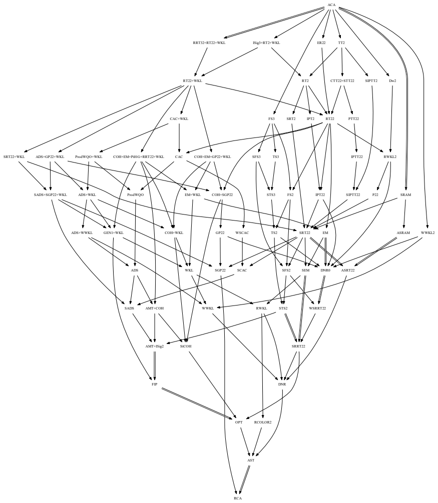
1. **The Zoo is useful.** Detailed citations, backed by a specialized inference engine for tracking knowledge.
2. **The Zoo works!** Detailed bibliography already assembled, producing authoritative results.
3. **The Zoo needs help.** Assembling the bibliography takes time, and requires *some* context. Contributions welcome!

Making Sense out of Chaos — Hopefully



The Pattern So Far

So far, every atypical principle between ACA_0 and RCA_0 falls on one of four “branches”:

- Ramsey theory (RT^2_2, ADS, etc.)
- Genericity (Π^0_1G [aka W2-GEN], FIP, etc.)
- Randomness (WWKL $\equiv$ 1-RAN, DNR, etc.)
- Compactness (WKL_0)

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Theorem (Seetapun and Slaman (1995))

$\mathsf{RCA}_0 + \mathsf{RT}^2_2 \not\models \mathsf{ACA}_0$.

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Theorem (Jockusch (1972))

RT^3_2 *is equivalent to* ACA_0 *over* RCA_0.

Genericity

Definition

X is n-generic if it is Cohen-generic for n-quantifier arithmetic.

Theorem (Posner (1-generic) and Jockusch (n-generic))

X is n-generic iff it meets or avoids every Σ^0_n collection of basic open sets.

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X is weakly $(n+1)$-generic if it meets all dense Σ^0_{n+1} sets.

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Genericity

Definition

- 1-GEN: For every X, there is a set 1-generic relative to X.
- Π^0_1G: roughly, the same for weak 2-generics.

Theorem (Hirschfeldt, Shore, and Slaman)

Π^0_1G *is equivalent to* AMT *over* $I\Sigma^0_2$.

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- 1-RAN: For every X, there is a set 1-random relative to X.
- n-RAN: For every X, there is a set random relative to $X^{(n-1)}$.

Randomness

- WWKL: Every computable binary tree with positive density at every level has an infinite path. ($\equiv$ 1-RAN over RCA_0)
- n-WWKL: Every $\emptyset^{(n-1)}$-computable binary tree...
($\equiv n\text{-RAN} + \mathrm{B}\Sigma^0_n$)

Theorem (Yu and Simpson (1990))

n-WWKL *does not imply WKL.*

Theorem (Avigad, Dean, and Rute (2012))

For $n > 1$, WKL does not imply n-WWKL.

Randomness

Theorem (Conidis & Slaman, analyzing Csima & Mileti)

2-RAN *implies* RRT^2_2 *over* RCA_0.

Definition

RRT^2_2: If $f : [\omega]^2 \to \omega$ has $|f^{-1}(c)| \leq 2$ for all c, then $f \restriction [R]^2$ is injective for some infinite R.

Typicality

Fix a notion of negligibility on 2^ω (a σ-ideal).

Definition

X is *typical* if it avoids all negligible classes with an effective description.

- n-random: measure 0, uniformly Σ^0_n description
- weakly $(n+1)$-generic: meager, Σ^0_n description

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A slight lie: technically, weakly n-generic avoids co-dense sets.
Thus, *a priori*, weakly n-generics are more than typical for this.

Typicality in Reverse Math

Existence of sufficiently typical sets has *some* power.

- 2-RAN implies RRT^2_2...
- Π^0_1G implies AMT...

But what are the limits?

Typicality in Reverse Math

Existence of sufficiently typical sets has *some* power.

- 2-RAN implies RRT^2_2...
- $\Pi^0_1\mathrm{G}$ implies AMT...

But what are the limits?

How much changes if we have a random oracle?

Assumptions

Fix a notion of negligibility in 2^ω. (σ-ideal)

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One more assumption:

If S is negligible, so is
$\{X : X \oplus Z \in S$ for non-negligibly many Z's$\}$.

- True for measure 0: Fubini's theorem
- True for first category: weak direction of Kuratowski-Ulam

Bounding Typicality

P and Q Π^1_2 principles:

$$(\forall I)[\Phi(I) \Rightarrow (\exists S)\Psi(I, S)]$$

Definition

P is *frequently solved* if every P-instance has P-solutions below non-negligibly many oracles.
A Q-instance I is *typically unsolved* if it has Q-solutions below negligibly many oracles.

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Theorem

If P is frequently solved and Q has a typically-unsolved instance, then there is an ω-model of $\mathrm{RCA}_0 + P + \neg Q$.

Consequences

n-RAN cannot imply:

- Π^0_1G, or AMT (same ω-models)
- RRT^3_2 (even though 2-RAN implies RRT^2_2)
- SEM (stable Erdös-Moser:
  every stable tournament has a transitive subtournament)

In fact. . . it cannot imply most of the Zoo!
Similar consequences for *n*-GEN; more work to be done.

Proof Idea

- We add solutions to P, while avoiding anything that makes solutions to the typically-unsolved instance I non-negligible.
- To do so, we add a solution X_1 not solving I...
  then a solution X_2 with $X_1 \oplus X_2$ not solving I...
- Actually, we avoid the closure of solutions to I under the operator:

$$B(C) = \{X : X \oplus Z \in C \text{ for non-negligibly many } Z\text{'s}\}.$$

- Still negligible, still closed upwards wrt I-computability.

Outside of Problem-Solution...

Hope: "frequently solved" means "true in a typical ω-model"...
No such luck.

"true in a typical ω-model" means "$(\omega, I(X)) \models P$ for typical X", where $I(X)$ is the Turing ideal generated by $\{X^{[i]}\}$.

Definition

If typical means measure 1, P does not have *the NRA property*.
(The NRA property corresponds to "typically-unsolved instance".)

The NRA Property

Definition

P does not have *NRA* if $(\omega, I(X)) \models P$ for almost all X.

If P is Π^1_2 (and typical means measure 1), equivalent to:
Almost every X bounds only P-instances solved with measure 1.

Stricter than "computable instances are frequently solved",
weaker than "all instances are frequently solved".

Theorem

If P does not have NRA, but Q does,
there is an ω-model of $\mathrm{RCA}_0 + P + \neg Q$.

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Thank you!